VECTOR BUNDLES ON PROJECTIVE VARIETIES WHICH SPLIT ALONG q-AMPLE SUBVARIETIES

MIHAI HALIC

Abstract. Given a vector bundle on a complex, smooth projective variety, I prove that its splitting along a very general, q-ample subvariety (for appropriate q), which admits sufficiently many embedded deformations implies the global splitting. The result goes significantly beyond the previously known splitting criteria obtained by restricting vector bundles to subvarieties.

I discuss the particular case of zero loci of sections in globally generated vector bundles, on one hand, and of sources of multiplicative group actions (corresponding to Bialynicki-Birula decompositions), on the other hand. Finally, I elaborate on the symplectic and orthogonal Grassmannians; I prove that the splitting of an arbitrary vector bundle on them can be read off from its restriction to a low dimensional 'sub'-Grassmannian.

Introduction

For a vector bundle $\mathscr{V}$ on the irreducible projective variety X, I consider its restriction $\mathscr{V}_Y$ to an irreducible subvariety $Y \subset X$, and investigate the following:

Question. Assuming that $\mathscr{V}_Y$ splits, under what assumptions on Y does $\mathscr{V}$ split too?

To my knowledge, *there is not a single reference which addresses the question in this generality.* Yet, the problem is interesting because it allows to probe the splitting of vector bundles on (high dimensional) varieties by restricting them to (possibly low dimensional) subvarieties. The goal of this article is to give a tentative answer to this question.

Horrocks' criterion [16] is the most widely known result of this type. It was extended in [5], for restrictions to ample divisors on (suitable) varieties. Surprisingly, restrictions to higher codimensional subvarieties have not been considered yet. For this reason, I studied in [12] the case when Y is the zero locus of a regular section in a globally generated, *ample* vector bundle $\mathcal{N}$ on X.

Unfortunately, this setting discards very basic situations, *e.g.* X is a product $X' \times V$ and $\mathcal{N} = \mathcal{O}_{X'}(1)$. Second, in perfect analogy with Horrocks' criterion, I proved that the splitting of a vector bundle on any Grassmannian can be verified by restricting it to an arbitrary, standardly embedded Grass(2; 4); this is *far from* being an *ample* subvariety. These observations led to consider 'sufficiently positive' subvarieties of X, which posses 'many' embedded deformations. Loosely speaking, the main result of this article is the following:

Theorem. *Let $\mathscr{V}$ be an arbitrary vector bundle on a projective variety X, defined over an uncountable, algebraically closed field of characteristic zero. Let Y be a subvariety of X satisfying the following properties:*

– *Y is q-ample for appropriate q (e.g. $q+1 \leqslant \dim X - 3\mathrm{codim}_X(Y)$, but not necessarily);*

2010 *Mathematics Subject Classification.* Primary 14J60; Secondary 14C25, 14M17, 14M15.

Key words and phrases. splitting criteria; vector bundles; q-ample subvarieties; Grassmannians.

– *The embedded deformations of Y cover an open subset of X, and their intersection pattern is sufficiently non-trivial.*

Then $\mathscr{V}$ splits if and only if it splits along a very general *deformation of Y.*

The precise statement can be found in theorem 2.8; applications to the case of globally generated vector bundles and of multiplicative group actions, with emphasis on homogeneous varieties, are stated in the theorems 3.7 and 5.3 respectively.

A central notion in this article is that of a $p^{>0}/q$*-ample subvariety* of a projective variety (cf. Definition 1.3); the definition is inspired from [23, 26, 2], also [9]. The reference [23] studies in depth the geometric properties of the 0-ample subvarieties. Although the generalization from 0- to q-ample is straightforward, it seems that here is the *first place* where this weaker property is considered and indeed used for concrete applications. A number of geometric properties of q-ample subvarieties and also criteria for q-ampleness, which are necessary for the splitting criterion above, are proved in the first section of the article.

The q-ampleness condition is asymptotic in nature, involving thickenings of subvarieties. For this reason, the price to pay for the generality of the theorem is that of testing the splitting along subvarieties which are *very general* within their own deformation space. Thus the q-ample case studied here can be characterized as probabilistic, in contrast with the case of ample vector bundles studied in [12], which is deterministic. Although the result is formulated algebraically, the proof uses complex analytic techniques; the final statement is deduced by base change.

The *essential feature* of the theorem is that of being *intrinsic* to the subvariety; it avoids additional data. This allows to treat in a *unified way* examples arising in totally different situations:

– zero loci of globally generated (not necessarily ample) vector bundles (cf. section 3), on one hand;

– homogeneous subvarieties of homogeneous varieties (cf. section 5), on the other hand.

As a by-product, we obtain several examples of q-ample subvarieties which are not zero loci of regular sections in globally generated q-ample vector bundles.

The case of isotropic Grassmannians is detailed in section 6. As I already mentioned, a vector bundle on $\mathrm{Gr}(u;w)$, $u\geqslant 2$, $w\geqslant u+2$, splits if and only if its restriction to *any* embedded $\mathrm{Gr}(2;4)$ splits. Unfortunately, the same proof breaks down in the case of the symplectic and orthogonal Grassmannians. However, our 'probabilistic' approach works very well.

Theorem. *Let $\mathscr{V}$ be a vector bundle either on the symplectic Grassmannian* $\text{sp-Gr}(u;w)$*, with $u\geqslant 2, w\geqslant 2u$, or the orthogonal Grassmannian* $\text{o-Gr}(u;w)$*, with $u\geqslant 3, w\geqslant 2u$. Then $\mathscr{V}$ splits if and only if it splits along a* very generally *embedded:*

– $\text{sp-Gr}(2;4)$, *in the symplectic case;*

– $$\left.\begin{array}{ll}\bullet\ \text{o-Gr}(3;6), & \textit{for } w=2u,\\ \bullet\ \text{o-Gr}(3;7), & \textit{for } w=2u+1,\\ \bullet\ \text{o-Gr}(3;8), & \textit{for } w\geqslant 2u+2,\end{array}\right\}\ \textit{in the orthogonal case.}$$

This dramatically simplifies the problem of deciding the splitting of vector bundles on Grassmannians. One should compare it with the cohomological criteria [22, 19, 3, 18], which involve a large number of vanishings.

Throughout this article, X stands for a smooth, projective, irreducible variety over $\mathbb{C}$, except a few statements involving base change arguments. We consider a vector bundle $\mathscr{V}$ on

X of rank r, and denote by $\mathscr{E} := \mathcal{E}nd(\mathscr{V})$ the endomorphisms of $\mathscr{V}$. For a subscheme $S \subset X$, we let $\mathscr{V}_S := \mathscr{V} \otimes \mathcal{O}_S$, *etc.*

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PART I: $p^{>0}/q$-AMPLE SUBVARIETIES

0. PROLOGUE: WHY q-AMPLENESS?

Before introducing the relevant definitions, I briefly explain the reasons which led to consider the notion of $p^{>0}/q$-ample subvarieties in the context of split vector bundles. One says that $\mathscr{V}$ splits on X if

$$\mathscr{V} = \bigoplus_{j\in J} \mathcal{L}_j \otimes \mathbb{C}^{m_j} \text{ with } \mathcal{L}_j \in \mathrm{Pic}(X) \text{ pairwise non-isomorphic, and } \sum_{j\in J} m_j = r = \mathrm{rk}(\mathscr{V}).$$

It is easy to see that $\mathscr{V}$ splits if and only if it admits an endomorphism with r distinct eigenvalues. The main tool to attack the splitting problem is the following observation.

Lemma 0.1 *Let $S \subset X$ be a connected, projective subscheme, such that $\mathscr{V}_S$ splits. If the restriction homomorphism* $\mathrm{res}_S : \Gamma(X, \mathscr{E}) \to \Gamma(S, \mathscr{E}_S)$ *is surjective, then $\mathscr{V}$ splits too.*

The subschemes considered in this article will usually be thickenings of subvarieties of X.

Definition 0.2 For $m \geqslant 0$, the m-th order thickening Y_m of a subvariety $Y \subset X$ is the closed subscheme defined by the sheaf of ideals $\mathcal{I}_Y^{m+1}$; with this convention, $Y_0 = Y$. The structure sheaves of two consecutive thickenings of Y fit into the exact sequence

$$0 \to \mathcal{I}_Y^m/\mathcal{I}_Y^{m+1} \to \mathcal{O}_{Y_m} \to \mathcal{O}_{Y_{m-1}} \to 0. \tag{0.1}$$

The *formal completion* of X along Y is defined as $\varinjlim Y_m$, and is denoted by $\hat{X}_Y$.

To apply lemma 0.1, I adopt a two-step strategy:

(i) Prove that $\Gamma(X, \mathscr{E}) \to \Gamma(\hat{X}_Y, \mathscr{E}_{\hat{X}_Y})$ is surjective; this step requires the $(\dim Y - 1)$-ampleness of Y. (Actually, we will consider also more general thickenings.)

(ii) Prove that $\Gamma(\hat{X}_Y, \mathscr{E}_{\hat{X}_Y}) \to \Gamma(Y, \mathscr{E}_Y)$ is surjective. This step is more involved, requires less ampleness (more positivity) of Y, and also its genericity in the space of embedded deformations.

If Y has the property that for all vector bundles $\mathcal{F}$ on X holds $H^1(X, \mathcal{F} \otimes \mathcal{I}_Y^m) = 0$ for all m sufficiently large (depending on $\mathcal{F}$), then the splitting of $\mathscr{V}$ along a high order thickening of Y implies the splitting on X. Indeed, simply take $\mathcal{F} = \mathscr{E}$. (For ample subvarieties, explicit lower bounds for m are obtained in [12].)

Lemma 0.3 *Assume that Y is $(\dim Y - 1)$-ample (cf. 1.2 below). Then $\mathscr{V}$ splits on X if and only if it does so along the formal completion of X along Y.*

The difficulty consists in proving that the splitting of $\mathscr{V}$ along a very general subvariety $Y \subset X$ implies the splitting of $\mathscr{V}$ along $\hat{X}_Y$. This step constitutes the body of the article.

1. Definition and properties of $p^{>0}/q$-ample subvarieties

1.1. **Definition and first properties.** The concept of q-ampleness for globally generated vector bundles was introduced in [24]; the q-ampleness in defined intrinsically in [2, 26] through cohomology vanishing properties. We recall the latter definition, the case of globally generated vector bundles being detailed in section 3.

Definition 1.1 (cf. [26, Theorem 7.1], [2, Lemma 2.1]) A *line bundle* $\mathcal{L}$ on a (reduced) projective Gorenstein variety $\tilde{X}$ is called *$\tilde{q}$-ample* if for any coherent sheaf $\tilde{\mathcal{F}}$ on $\tilde{X}$ holds:

$$H^t(\tilde{\mathcal{F}} \otimes \mathcal{L}^m) = 0,\ \forall\, t > \tilde{q},\ \forall\, m \gg 0.$$

We say that a *vector bundle* $\mathcal{N}$ on X is *q-ample* if $\mathcal{O}_{\mathbb{P}(\mathcal{N}^\vee)}(1)$ on $\mathbb{P}(\mathcal{N}^\vee)$ is q-ample.

Proposition 1.2 (cf. [26, Section 7], [23, Definition 3.1]). *Let $Y \subset X$ be an equidimensional subscheme of codimension δ, $\tilde{X} := \mathrm{Bl}_Y(X)$ be the blow-up of the ideal of Y, and $E_Y \subset \tilde{X}$ be the exceptional divisor. Assume that $\tilde{X}$ is Gorenstein. The following statements are equivalent:*

(i) *For any locally free (hence also for any coherent) sheaf $\tilde{\mathcal{F}}$ on $\tilde{X}$ holds*

$$H^t(\tilde{X}, \tilde{\mathcal{F}} \otimes \mathcal{O}_{\tilde{X}}(mE_Y)) = 0,\ \forall\, t \geqslant q+\delta,\ \forall\, m \gg 0, \quad \text{(that is, } E_Y \text{ is } (q+\delta-1)\text{-ample)}; \tag{1.1}$$

(ii) *For all locally free sheaves (that is vector bundles) $\mathcal{F}$ on X holds*

$$H^t(X, \mathcal{F} \otimes \mathcal{I}_Y^m) = 0,\ \forall\, t \leqslant \dim Y - q,\ \forall\, m \gg 0. \tag{1.2}$$

Proof. Apply the Serre duality on $\tilde{X}$. □

If X is smooth and $Y \subset X$ is a locally complete intersection (*lci* for short), then $\tilde{X}$ is automatically Gorenstein. We are primarily interested in the cohomology vanishing property (1.2); for this reason, we introduce an *ad hoc* terminology.

Definition 1.3 We say that a lci subvariety $Y \subset X$ *is (has the property)* $p^{>0}$ if for any vector bundle $\mathcal{F}$ on X holds:

$$\exists\, m_{\mathcal{F}} \geqslant 1,\ \forall\, m \geqslant m_{\mathcal{F}},\ \forall\, t \leqslant p, \quad H^t(X, \mathcal{F} \otimes \mathcal{I}_Y^{m+1}) = 0. \tag{1.3}$$

Intuitively, Y is $p^{>0}$ if its normal bundle at each point contains at least p 'positive' directions. Probably the appropriate name for this property of Y would be '*q-ample subvariety*', with $q = \dim Y - p$. (The case of ample subvarieties [23] corresponds to $q = 0$.) The terminology in the definition is made to emphasize the amount of positivity of the various objects which appear subsequently.

Proposition 1.4 (i) *For $Y \subset X$ irreducible, lci holds:*

$$Y \text{ is } p^{>0} \quad \Leftrightarrow \quad \begin{cases} \text{the normal bundle } \mathcal{N} = \mathcal{N}_{Y/X} \text{ is } (\dim Y - p)\text{-ample,} \\ \text{the cohomological dimension } \operatorname{cd}(X \setminus Y) \leqslant \dim X - (p+1). \end{cases} \tag{1.4}$$

(Recall that $\operatorname{cd}(X \setminus Y) \geqslant \operatorname{codim}(Y) - 1$ for any closed subvariety.)
(ii) *Let $Z \subset Y$ is $p^{>0}$, $Y \subset X$ is $r^{>0}$, and are both irreducible lci, then*

$$\begin{cases} \mathcal{N}_{Z/X} \text{ is } \big(\dim Y + \dim Z - (r+p)\big)\text{-ample,} \\ \operatorname{cd}(X \setminus Z) \leqslant \dim X - (\min\{r,p\} + 1). \end{cases} \tag{1.5}$$

In particular, $Z \subset X$ is $\big(p - (\dim Y - r)\big)^{>0}$.

For the definition and the properties of the cohomological dimension of a variety, the reader may consult [15, Ch. III, §3].

Proof. (i) ($\Rightarrow$) Let $\mathcal{F}$ be a vector bundle on X. Since $E_Y = \mathbb{P}(\mathcal{N})$, the exact sequence

$$0 \to \mathcal{O}_{\tilde{X}}((m-1)E_Y) \to \mathcal{O}_{\tilde{X}}(mE_Y) \to \mathcal{O}_{\mathbb{P}(\mathcal{N})}(-m) \to 0 \tag{1.6}$$

implies $H^t(\mathbb{P}(\mathcal{N}), \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{N})}(-m)) = 0$, for all $t \geqslant \dim X - p$ and $m \gg 0$. Let us denote $\delta := \operatorname{codim}_X(Y)$; the equality

$$H^{\delta+t}(\mathbb{P}(\mathcal{N}), \mathcal{O}_{\mathbb{P}(\mathcal{N})}(-\delta - m) \otimes (\mathcal{F} \otimes \det(\mathcal{N})^{-1})) = H^{t+1}(\mathbb{P}(\mathcal{N}^\vee), \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{N}^\vee)}(m)), \tag{1.7}$$

yields $H^{t+1}(\mathbb{P}(\mathcal{N}^\vee), \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{N}^\vee)}(m)) = 0$, for all $t \geqslant \dim Y - p$. The second implication is [23, Proposition 5.1], as E_Y is $\tilde{q} = \delta + (\dim Y - p) - 1$ ample.
($\Leftarrow$) The $(\dim Y - p)$-ampleness of $\mathcal{O}_{\mathbb{P}(\mathcal{N}^\vee)}(1)$, combined with (1.6), implies

$$H^t(\mathcal{F} \otimes \mathcal{O}_{\tilde{X}}((m-1)E_Y)) \overset{\cong}{\to} H^t(\mathcal{F} \otimes \mathcal{O}_{\tilde{X}}(mE_Y)), \text{ for } t \geqslant \dim X - p \text{ and } m \gg 0,$$

so $H^t(\mathcal{F} \otimes \mathcal{O}_{\tilde{X}}(mE_Y)) = \varinjlim_k H^t(\mathcal{F} \otimes \mathcal{O}_{\tilde{X}}(kE_Y))$; according to [23, (5.1)], the right hand side equals $H^t(X \setminus Y, \mathcal{F})$, which vanishes.
(ii) In the sequence

$$0 \to \mathcal{N}_{Z/Y} \to \mathcal{N}_{Z/X} \to \mathcal{N}_{Y/X}\big|_Z \to 0,$$

the extremities are $(\dim Z - p)$, respectively $(\dim Y - r)$-ample; the subadditivity of the amplitude [2, Theorem 3.1] yields the conclusion. The bound on the cohomological dimension follows by repeating *ad litteram* [23, Proposition 6.4]; for the comfort of the reader, we recall the proof here. Let $U_Z := X \setminus Z, U_Y := X \setminus Y$ and consider an arbitrary sheaf $\mathcal{F}$ on X. In the exact sequence

$$\ldots \to H^i_{Y \setminus Z}(U_Z, \mathcal{F}) \to H^i(U_Z, \mathcal{F}) \to H^i(U_Y, \mathcal{F}) \to \ldots,$$

the right hand side vanishes for $i \geqslant \dim X - r$, because $Y \subset X$ is $r^{>0}$. We claim that the left hand side vanishes too, for $i \geqslant \dim X - p$. Indeed, it can be computed by using the spectral sequence (cf. [11, Exposé I, Théorème 2.6]):

$$H^b(U_Z, \mathcal{H}^a_{Y \setminus Z}(\mathcal{F})) \;\Rightarrow\; H^{a+b}_{Y \setminus Z}(U_Z, \mathcal{F}),$$

where $\mathcal{H}^a_{Y \setminus Z}(\mathcal{F})$ stands for the local cohomology sheaf, with support on $Y \setminus Z$. The term on the left has the following two properties: first, $\mathcal{H}^a_{Y \setminus Z}(\mathcal{F}) = \varinjlim_m \mathcal{E}xt^a(\mathcal{O}_{U_Z}/\mathcal{I}^m_{Y \setminus Z}, \mathcal{F})$ (cf. [11, Exposé II, Théorème 2]), the $\mathcal{E}xt$ groups are supported on $Y \setminus Z$, and $Z \subset Y$ is $p^{>0}$, all together imply that

$$H^b(U_Z, \mathcal{H}^a_{Y\setminus Z}(\mathcal{F})) = 0,\ \forall\, b \geqslant \dim Y - p;$$

second, $\mathcal{H}^a_{Y\setminus Z}(\mathcal{F}) = 0$, $\forall\, a \geqslant \dim X - \dim Y + 1$, because $Y \subset X$ is lci. All together, we deduce that $H^i_{Y\setminus Z}(U_Z, \mathcal{F}) = 0$, for $i \geqslant (\dim X - \dim Y) + (\dim Y - p - 1) + 1$. For the final statement, observe that Z satisfies the conditions (i). □

Remark 1.5 We recall from [15, Ch. III, Corollary 3.9] the following: if X is Gorenstein and $Y \subset X$ is a closed subvariety such that $\mathrm{cd}(X \setminus Y) \leqslant \dim X - 2$, then Y is connected.[1]

In particular, if $Y \subset X$ is lci and $1^{>0}$, then it is connected, equidimensional. (This is straightforward: take $\mathcal{F} = \mathcal{O}_X$ in (1.3).) In particular, if Y is smooth then it is irreducible.

Example 1.6 (an unexpected behaviour) The example shows that in (1.4) both conditions are necessary. For $X := \mathbb{P}^n \times \mathbb{P}^n$ and $Y :=$ the diagonal $\cong \mathbb{P}^n$, the following hold:

(i) $\mathcal{N}_{Y/X} \cong T_{\mathbb{P}^n}$, so the normal bundle is ample and globally generated;

(ii) The morphism $X \setminus Y \to \mathrm{Flag}(1, 2; \mathbb{C}^{n+1})$, $(x_1, x_2) \mapsto \big(x_1, \langle x_1, x_2\rangle\big)$, is an $\mathbb{A}^1$-fibre bundle, hence $\mathrm{cd}(X \setminus Y) = 2n - 1$ and Y is only $0^{>0}$ in X.

The lack of sufficient positivity of the normal bundle $\mathcal{N}_{Z/X}$ prevents to conclude that $Z \subset X$ is $\min\{r, p\}^{>0}$. However, the next proposition shows that it is close to be so.

Definition 1.7 (i) For any vector bundle $\mathcal{F}$ on X and a closed subscheme $Z \subset X$, define

$$\tilde{H}^t(Z, \mathcal{F}) := \left\{ h \in H^t(Z, \mathcal{F}) \;\middle|\; \begin{array}{l} \exists\, \mathcal{U} \supset Z \text{ open subset of } X, \\ \exists\, \tilde{\alpha} \in H^t(\mathcal{U}, \mathcal{F}) \text{ such that } \alpha = \mathrm{res}^{\mathcal{U}}_Z(\tilde{\alpha}) \end{array} \right\}.$$

Here $\mathcal{U}$ can be *either* a *Zariski or* an *analytic* (tubular) open neighbourhood of Z.

(ii) We say that a subscheme $Z \subset X$ is $p^{\gtrsim 0}$ (*approximately* $p^{>0}$) if there is a decreasing sequence of sheaves of ideals $\{\mathcal{J}_m\}_m$ such that the following hold:

$$\begin{array}{l} \bullet\ \ \forall\, m, n \geqslant 1\ \exists\, m' > m,\ n' > n \text{ such that } \mathcal{J}_{m'} \subset \mathcal{I}^m_Y,\ \mathcal{I}^{n'}_Y \subset \mathcal{J}_n; \\ \bullet\ \ \text{for any vector bundle } \mathcal{F} \text{ on } X, \\ \quad\ \exists\, m_{\mathcal{F}} \geqslant 1 \text{ such that } H^t\big(X, \mathcal{F} \otimes \mathcal{J}_m\big) = 0,\ \forall\, m \geqslant m_{\mathcal{F}}\ \forall\, t \leqslant p. \end{array} \tag{1.8}$$

Proposition 1.8 *If $Z \subset Y$, $Y \subset X$ be lci, and both $p^{>0}$, then $Z \subset X$ is $p^{\gtrsim 0}$. In particular, for all vector bundles $\mathcal{F}$ on X holds:*

$$\mathrm{res}^X_Z : H^t(X, \mathcal{F}) \to \tilde{H}^t(Z, \mathcal{F}) \text{ is: } \begin{cases} \text{- an isomorphism, for } t \leqslant p - 1, \\ \text{- injective, for } t = p. \end{cases}$$

Proof. Since $Z \subset Y \subset X$ are lci, for any $l \geqslant a$ one has the exact sequence:

$$0 \to \frac{\mathcal{I}^a_Y}{\mathcal{I}^{a+1}_Y} \otimes \left(\frac{\mathcal{I}_Z}{\mathcal{I}_Y}\right)^{l-a} \to \frac{\mathcal{I}^l_Z + \mathcal{I}^{a+1}_Y}{\mathcal{I}^{a+1}_Y} \to \frac{\mathcal{I}^l_Z + \mathcal{I}^a_Y}{\mathcal{I}^a_Y} \to 0. \tag{1.9}$$

[1] In *loc. cit.* the result is stated for X smooth. However, this assumption is used only to apply the Serre duality; so the same proof works in the Gorenstein case too. We are interested in this generalization for X an lci variety, to avoid additional transversality assumptions.

Indeed, just use $(\mathcal{I}_Z^l+\mathcal{I}_Y^a)/\mathcal{I}_Y^a \cong \mathcal{I}_Z^l/\mathcal{I}_Y^a\cdot\mathcal{I}_Z^{l-a}$. Note that the left hand side is an $\mathcal{O}_Y$-module; $\mathcal{I}_Z/\mathcal{I}_Y=\mathcal{I}_{Z\subset Y}$ is the ideal of $Z\subset Y$, and that $\mathcal{I}_Y^a/\mathcal{I}_Y^{a+1}=\mathrm{Sym}^a\,\mathcal{N}_{Y/X}^\vee$ is the symmetric power of co-normal bundle. The $p^{>0}$ assumption implies that there are integers $k_{\mathcal{F}}, l_{\mathcal{F}}$, and a linear function $l(k)=\lambda k+\mu$ (λ,μ independent of $\mathcal{F}$) with the following properties:

$$\begin{aligned} H^t(\mathcal{F}\otimes\mathcal{I}_Y^k)&=0, \quad \forall\, t\leqslant p,\ \forall\, k\geqslant k_{\mathcal{F}},\\ H^t(\mathcal{F}_Y\otimes\mathcal{I}_{Z\subset Y}^l)&=0, \quad \forall\, t\leqslant p,\ \forall\, l\geqslant l_{\mathcal{F}},\\ H^t(\mathcal{F}_Y\otimes\mathrm{Sym}^a\,\mathcal{N}_{Y/X}^\vee\otimes\mathcal{I}_{Z\subset Y}^{l-a})&=0, \quad \forall\, t\leqslant p,\ \forall\, a\leqslant k,\ \forall\, l\geqslant l(k). \end{aligned}$$

The last claim uses the uniform q-ampleness property [26, Theorem 6.4] and the subadditivity of the regularity [26, Theorem 3.4]: first, there is a linear function $l(r)$ such that for any vector bundle $\mathcal{F}$ with regularity $\mathrm{reg}(\mathcal{F}_Y)\leqslant r$ holds

$$H^t(\mathcal{F}_Y\otimes\mathcal{I}_{Z\subset Y}^l)=0, \text{ for } t\leqslant p,\ l\geqslant l(r);$$

second, for $a\leqslant k$, $\mathrm{reg}(\mathcal{F}_Y\otimes\mathrm{Sym}^a\,\mathcal{N}_{Y/X}^\vee)\leqslant\mathrm{linear}(k)$. Recursively for $a=1,\dots,k$, starting with $\frac{\mathcal{I}_Z^l+\mathcal{I}_Y}{\mathcal{I}_Y}=\mathcal{I}_{Z\subset Y}^l$, the exact sequence (1.9) yields:

$$H^t\Big(\mathcal{F}\otimes\frac{\mathcal{I}_Z^l+\mathcal{I}_Y^k}{\mathcal{I}_Y^k}\Big)=0,\ \forall\, l\geqslant l(k).$$

Now plug this into $0\to\mathcal{I}_Y^k\to\mathcal{I}_Z^l+\mathcal{I}_Y^k\to\frac{\mathcal{I}_Z^l+\mathcal{I}_Y^k}{\mathcal{I}_Y^k}\to 0$ (tensored by $\mathcal{F}$), and deduce that

$$H^t\big(\mathcal{F}\otimes(\mathcal{I}_Y^k+\mathcal{I}_Z^l)\big)=0,\ \forall\, t\leqslant p,\ \ \forall\, k\geqslant k_{\mathcal{F}},\ \forall\, l\geqslant l(k).$$

We denote by $Z_{l,k}$ the subscheme defined by $\mathcal{I}_Y^k+\mathcal{I}_Z^l$; it is an 'asymmetric' thickening of Z in X. For any l as above, there is $m>l$ such that

$$\mathcal{I}_Y^l+\mathcal{I}_Z^m\subset\mathcal{I}_Z^l\subset\mathcal{I}_Y^k+\mathcal{I}_Z^l\ \Rightarrow\ Z_{l,k}\subset Z_l\subset Z_{m,l}.$$

For $m>l>k$ as above, one has the commutative diagram

$$\begin{array}{ccccc} & \overset{\mathrm{res}^X_{m,l}}{\nearrow} & H^t(\mathcal{F}_{Z_{m,l}}) & \overset{\mathrm{res}^{m,l}_l}{\searrow} & \\ H^t(\mathcal{F}) & & \xrightarrow{\qquad\xi\qquad} & & H^t(\mathcal{F}_{Z_l}). \\ & \underset{\mathrm{res}^X_{l,k}}{\searrow} & H^t(\mathcal{F}_{Z_{l,k}}) & \underset{\mathrm{res}^l_{l,k}}{\swarrow} & \end{array}$$

Notice that $\mathrm{res}^l_{l,k}\circ\xi=\mathrm{res}^X_{l,k}$, which is injective, so ξ is injective. It remains to prove that ξ maps onto $\tilde{H}^t(\mathcal{F}_{Z_l})$, for $t\leqslant p-1$: assume $\alpha=\mathrm{res}^{\mathcal{U}}_{Y_l}(\tilde\alpha)$, with $\mathcal{U}\supset Z$ open and $\tilde\alpha\in H^t(\mathcal{F}_{\mathcal{U}})$; then $\alpha=\mathrm{res}^m_l(\,\mathrm{res}^{\mathcal{U}}_{m,l}(\tilde\alpha)\,)$, and $\mathrm{res}^{\mathcal{U}}_{m,l}(\tilde\alpha)$ belongs to the image of $\mathrm{res}^X_{m,l}$. □

Lemma 1.9 *Let $\varphi: X\to X'$ be a flat, surjective morphism, with X, X' smooth, whose fibres are d-dimensional. If $Y'\subset X'$ is lci and $p^{>0}$, then $Y=\varphi^{-1}(Y')\subset X$ is the same.*

Proof. Since f is flat, Y is still lci in X and $\mathrm{codim}_X(Y)=\mathrm{codim}_{X'}(Y')=\delta$. Let us check the property (1.1). The universality property of the blow-up (cf. [14, Ch. II, Corollary 7.15]) yields the commutative diagram

$$\begin{array}{ccc} \tilde X=\mathrm{Bl}_Y(X) & \xrightarrow{\ \tilde\varphi\ } & \tilde X'=\mathrm{Bl}_{Y'}(X) \\ \downarrow & & \downarrow \\ X & \xrightarrow{\ \varphi\ } & X'. \end{array}$$

The morphism $\tilde{\varphi}$ still has d-dimensional fibres and $\tilde{\varphi}^*\mathcal{O}_{\tilde{X}'}(E_{Y'}) = \mathcal{O}_{\tilde{X}}(E_Y)$. For any coherent sheaf $\tilde{\mathcal{F}}$ on $\tilde{X}$ holds:

$$R^i\tilde{\varphi}_*(\tilde{\mathcal{F}} \otimes \mathcal{O}_{\tilde{X}}(mE_Y)) = R^i\varphi_*\tilde{\mathcal{F}} \otimes \mathcal{O}_{\tilde{X}'}(mE_{Y'}), \quad R^{>d}\tilde{\varphi}_*\tilde{\mathcal{F}} = 0.$$

As $Y' \subset X'$ is $p^{>0}$, we deduce that

$$H^t\big(\tilde{X}', R^i\tilde{\varphi}_*\tilde{\mathcal{F}} \otimes \mathcal{O}_{\tilde{X}'}(mE_{Y'})\big) = 0, \text{ for } i = 0, \ldots, d,\ t \geqslant (\dim X' - p), \text{ and } m \gg 0.$$

Then the Leray spectral sequence implies that E_Y is $\big((\dim X' - p) + d - 1\big)$-ample. □

1.2. **Criterion for the positivity of a subvariety.** Many examples of q-ample subvarieties occur as zero loci of regular sections in (Sommese) q-ample vector bundles. (This will be detailed in the section 3.) However, it will be necessary to test the q-ampleness of a subvariety in more general circumstances.

Proposition 1.10 *Let the situation be as in 1.2. Assume that there is an irreducible variety V and a morphism $b : \tilde{X} \to V$ such that $\mathcal{O}_{\tilde{X}}(E_Y)$ is b-relatively ample. Then $Y \subset X$ is $p^{>0}$, for $p := \dim X - \dim b(\tilde{X}) - 1$.*
If $\tilde{X}$ is the blow-up of an ideal $\mathcal{J} \subset \mathcal{O}_X$ with $\sqrt{\mathcal{J}} = \mathcal{I}_Y$ and b is as above, then Y is $p^{\gtrsim 0}$.

Proof. Let $\tilde{\mathcal{F}}$ be a coherent sheaf on $\tilde{X}$, and $j \geqslant \dim X - p > \dim b(\tilde{X})$. Then

$$R^t b_*(\tilde{\mathcal{F}} \otimes \mathcal{O}_{\tilde{X}}(mE_Y)) = 0, \text{ for } t > 0 \text{ and } m \gg 0,$$

implies that $H^j\big(\tilde{X}, \tilde{\mathcal{F}} \otimes \mathcal{O}_{\tilde{X}}(mE_Y)\big) = H^j\big(V, b_*(\tilde{\mathcal{F}} \otimes \mathcal{O}_{\tilde{X}}(mE_Y))\big)$. But the right hand side vanishes, because $\operatorname{Supp} b_*(\tilde{\mathcal{F}} \otimes \mathcal{O}_{\tilde{X}}(mE_Y))$ is at most $\dim b(\tilde{X})$-dimensional. □

Remark 1.11 In the section 2, we will be interested in families of $p^{>0}$ subvarieties of X; the proposition generalizes straightforwardly. Let $\mathcal{Y} \subset S \times X$ be a smooth family of subvarieties of X indexed by some parameter space S. Assume that there is an S-variety V such that

$$\mathcal{O}(E_{\mathcal{Y}}) \text{ is relatively ample for a morphism } \mathrm{Bl}_{\mathcal{Y}}(S \times X) \to V.$$

Then Y_s is $(\dim(S \times X) - \dim V - 1)^{>0}$, for all $s \in S$ such that Y_s is lci.

The criterion applies to the situations discussed in sections 3 and 4, that is the case of zero sections in globally generated vector bundles, and of sources of G_m-actions.

1.3. **q-positive line bundles.** Intuitively, the hypotheses in 1.10 allow $\mathcal{O}_{\tilde{X}}(E_Y)$ to be 'negative' in at most $\dim b(\tilde{X}) < \dim X - p$ directions. As we will see below, this stronger positivity property allows to control $\mathrm{Pic}(Y)$.

Definition 1.12 (cf. [9]) A line bundle $\mathcal{L}$ on a smooth projective variety $\tilde{X}$ is *q-positive* if it admits a Hermitian metric whose curvature form has at most q negative (or zero) eigenvalues.

It is generally true (cf. [1, Proposition 28], [9, Proposition 2.1]) that a q-positive line bundle $\mathcal{L}$ satisfies $H^{\geqslant q+1}(\tilde{\mathcal{F}} \otimes \mathcal{L}^m) = 0$, for any vector bundle $\tilde{\mathcal{F}}$ and $m \gg 0$, that is

$$\mathcal{L} \text{ is } q\text{-positive} \ \Rightarrow\ \mathcal{L} \text{ is } q\text{-ample}.$$

Proposition 1.13 *Let the situation be as in proposition 1.10, with X, Y smooth. Then $\mathcal{O}_{\tilde{X}}(E_Y)$ is $\dim b(\tilde{X})$-positive.*

Proof. One may assume that V is smooth; otherwise, take an embedding into a smooth variety. As $\mathcal{O}_{\tilde{X}}(E_Y)$ is b-relatively ample, there is an embedding $\tilde{X} \xrightarrow{\iota} \mathbb{P}^N \times V$ (over V) such that $\mathcal{O}_{\tilde{X}}(m_0 E_Y) = \iota^*(\mathcal{O}_{\mathbb{P}^N}(1) \boxtimes \mathcal{M})$, for some $m_0 > 0$ and $\mathcal{M} \in \mathrm{Pic}(V)$. Take a strictly positive Hermitian metric on $\mathcal{O}_{\mathbb{P}^N}(1)$, an arbitrary on $\mathcal{M}$, and the product metric on $\mathcal{O}_{\mathbb{P}^N}(1) \boxtimes \mathcal{M}$.

The maximal rank of $\mathrm{d}b_{\tilde{x}}$ is $\dim b(\tilde{X})$, attained on a dense open subset of $\tilde{X}$, so

$$\mathrm{rk}(\mathrm{d}b_{\tilde{x}}) \leqslant \dim b(\tilde{X}),\ \forall\, \tilde{x} \in \tilde{X}.$$

Since ι is an embedding, the curvature of the pull-back metric on $\mathcal{O}_{\tilde{X}}(m_0 E_Y)$ is positive definite on $\mathrm{Ker}(\mathrm{d}b_{\tilde{x}})$, and $\dim \mathrm{Ker}(\mathrm{d}b_{\tilde{x}}) \geqslant \dim X - \dim b(\tilde{X})$. So, at each point of $\tilde{X}$, there are at most $\dim b(\tilde{X})$ negative eigenvectors. □

Lemma 1.14 *Let $\varphi : X \to X'$ be a smooth morphism of relative dimension d. If $\mathcal{L}'$ is a q-positive line bundle on X', then $\varphi^*\mathcal{L}'$ is $(q+d)$-positive on X.*

Proof. Take a metric in $\mathcal{L}'$ as in 1.12, and pull it back to $\mathcal{L}$. □

1.4. **Geometric properties of q-ample subvarieties.** In the section 2 we will be concerned with families of $p^{>0}$ subvarieties. The non-emptiness and connectedness of their intersections, as well as their Picard groups will be essential; these issues are investigated below.

1.4.1. *Non-emptiness and connectedness of the intersections.* For a family of irreducible subvarieties $\mathcal{Y} \subset S \times X$ of X indexed by some variety S, we denote

$$Y_{st} := Y_s \cap Y_t, \quad Y_{ost} := Y_o \cap Y_s \cap Y_t,\ \forall\, o, s, t \in S. \tag{1.10}$$

Lemma 1.15 *Let $\mathcal{Y} = \{Y_s\}_{s \in S}$ be a family of δ-codimensional, lci subvarieties of X such that their double and triple intersections are equidimensional lci (if non-empty). Assume that*

$$\mathrm{cd}(X \setminus Y_s) \leqslant \dim(X) - 2\delta - 2,\ \forall\, s \in S.$$

Then, for all $o, s, t \in S$, the intersections Y_{os} and Y_{ost} are indeed non-empty *and* connected. *In particular, the conclusion holds if $\mathcal{Y}$ is a family of $p^{>0}$ subvarieties of X, with $2\delta + 1 \leqslant p$.*

Proof. If $Y_{os} = \emptyset$, then $Y_s \subset X \setminus Y_o$ implies $\dim X - 2\delta - 2 \geqslant \mathrm{cd}(X \setminus Y_o) \geqslant \dim X - \delta$, a contradiction. Hence the double intersections are non-empty. If $Y_{ost} = \emptyset$, then $Y_{os} \subset Y_o \setminus Y_{ot}$, hence we obtain $\dim X - 2\delta - 2 \geqslant \mathrm{cd}(Y_o \setminus Y_{ot}) \geqslant \dim X - 2\delta$, another contradiction. The connectedness of the intersections follows from the remark 1.5:

$$\begin{aligned} &\mathrm{cd}(Y_o \setminus Y_{os}) \leqslant \mathrm{cd}(X \setminus Y_s) \leqslant \dim X - 2\delta - 2 < \dim Y_o - 2;\\ &\mathrm{cd}(Y_{os} \setminus Y_{ost}) \leqslant \mathrm{cd}(X \setminus Y_t) \leqslant \dim X - 2\delta - 2 \leqslant \dim Y_{os} - 2. \end{aligned}$$

The last claim follows from the proposition 1.4. □

1.4.2. *The Picard group of $p^{>0}$ subvarieties.* Let $Y \subset X$ be smooth. We are interested when the pull-back $\mathrm{res}^X_Y : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is an isomorphism.

Lemma 1.16 *Let $Y \subset X$ be a subvariety. Then $\mathrm{res}^X_Y : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is an isomorphism as soon as $H^t(X, \mathbb{Z}) \to H^t(Y; \mathbb{Z})$, $t = 1, 2$, are isomorphisms. In particular, res^X_Y is an isomorphism if $H_{2\dim_{\mathbb{C}} X - t}(X \setminus Y; \mathbb{Z}) = 0$ for $1 \leqslant t \leqslant 3$.*

Proof. The hypothesis implies $H^t(X;\mathbb{C}) \overset{\cong}{\to} H^t(X;\mathbb{C})$, for $t = 1, 2$; the Hodge decomposition yields $H^t(X;\mathcal{O}_X) \overset{\cong}{\to} H^t(X;\mathcal{O}_Y)$. Now compare the exponential sequences for X and Y:

$$\begin{CD} H^1(X;\mathbb{Z}) @>>> H^1(X;\mathcal{O}_X) @>>> \mathrm{Pic}(X) @>>> H^2(X;\mathbb{Z}) @>>> H^2(X;\mathcal{O}_X) \\ @V{\mathrm{res}^X_Y}VV @VVV @VVV @V{\mathrm{res}^X_Y}VV @VVV \\ H^1(Y;\mathbb{Z}) @>>> H^1(Y;\mathcal{O}_X) @>>> \mathrm{Pic}(Y) @>>> H^2(Y;\mathbb{Z}) @>>> H^2(Y;\mathcal{O}_X). \end{CD} \tag{1.11}$$

□

Remark 1.17 The restriction $H^t(X;\mathbb{Q}) \to H^t(Y;\mathbb{Q})$ is an isomorphism for $t \leqslant p-1$, for any $p^{>0}$ subvariety Y (cf. [23, Corollary 5.2]). Hence, if Y is $3^{>0}$, $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Y)$ is a finite morphism, and $\mathrm{NS}(X) \to \mathrm{NS}(Y)$ has finite index.

Theorem 1.18 *Let $Y \subset X$ be a smooth subvariety. Assume there is a smooth variety V and a morphism $\tilde{X} = \mathrm{Bl}_Y(X) \overset{b}{\to} V$ such that $\mathcal{O}_{\tilde{X}}(E_Y)$ is b-relatively ample. Then holds*

$$H^t(X;\mathbb{Z}) \overset{\cong}{\to} H^t(Y;\mathbb{Z}),\ \textit{for } t \leqslant p-1 \textit{ and } p := \dim X - \dim b(\tilde{X}) - 1.$$

Thus $\mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ *is an isomorphism, for $p \geqslant 3$. If Y' is a small, smooth deformation of Y, then* $\mathrm{Pic}(X) \to \mathrm{Pic}(Y')$ *is still an isomorphism, and Y' is $p^{>0}$ too.*

Proof. The hypothesis implies that $\mathcal{O}_{\tilde{X}}(E_Y)$ is $\dim b(\tilde{X})$-positive (cf. proposition 1.13). Then [7, Theorem III], [23, Lemma 10.1] imply that

$$H^t_c(X \setminus Y;\mathbb{Z}) = H^t_c(\tilde{X} \setminus E_Y;\mathbb{Z}) = 0,\ \forall\, t \leqslant \dim X - \dim b(\tilde{X}) - 1.$$

The long exact sequence in cohomology yields the conclusion. The second statement is the openness of the q-ample property [26, Theorem 8.1]. □

Part II: general results

2. The splitting criterion

Let $\mathscr{V} = \bigoplus_{j\in J} \mathcal{L}_j \otimes \mathbb{C}^{m_j}$ be a split vector bundle, with $\mathcal{L}_j \in \mathrm{Pic}(X)$ pairwise non-isomorphic. In this case, $\mathscr{V}_j := \mathcal{L}_j \otimes \mathbb{C}^{m_j}$, $j \in J$, are called the *isotypical components* of the splitting. If $\bigoplus_{j\in J} \mathcal{L}_j \otimes \mathbb{C}^{m_j}$ and $\bigoplus_{j'\in J'} \mathcal{L}'_{j'} \otimes \mathbb{C}^{m'_{j'}}$ are two splittings, then there is a bijective function $J \overset{\epsilon}{\to} J'$ such that $\mathcal{L}'_{\epsilon(j)} \cong \mathcal{L}_j$ and $m'_{\epsilon(j)} = m_j$ for all $j \in J$ (cf. [4, Theorem 1 and 2].) However, the isotypical components are *not uniquely defined*, because the global automorphisms of $\mathscr{V}$ send a splitting into a new one. We consider the partial order '$\prec$' on $\mathrm{Pic}(X)$:

$$\mathcal{L} \prec \mathcal{L}' \quad \text{if } \mathcal{L} \neq \mathcal{L}' \text{ and } \Gamma(X, \mathcal{L}^{-1}\mathcal{L}') \neq 0. \tag{2.1}$$

The isotypical components corresponding to the *maximal* elements, are *canonically defined.*

Lemma 2.1 *Let $M \subset J$ be the subset of maximal elements with respect to $\prec$. Then there is a natural, injective homomorphism of vector bundles*

$$\mathrm{ev}_M : \bigoplus_{j\in M} \mathcal{L}_j \otimes \Gamma(X, \mathcal{L}_j^{-1} \otimes \mathscr{V}) \to \mathscr{V}. \tag{2.2}$$

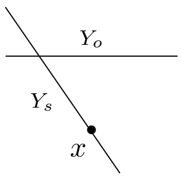
This is an essential observation because, given a family of subvarieties $\{Y_s\}_{s\in S}$ of X such that $\mathscr{V}$ splits along each of them, we can *glue together* the maximal isotypical components of $\mathscr{V}_{Y_s}$ (cf. lemma 2.6 below).

2.1. **Gluing of split vector bundles.** Let S be an irreducible quasi-affine variety, and $\mathcal{Y} \subset S \times X$ be subvariety (denote by π, ρ the morphisms to S and X, respectively) with the following properties:

(0) $\operatorname{Pic}(S)$ is trivial. (This can be achieved by shrinking S.)
(i) The morphism $\mathcal{Y} \xrightarrow{\pi} S$ is proper, smooth, with connected fibres.
(ii) For $o \in S$, define $S(o) := \{s \in S \mid Y_s \cap Y_o \neq \emptyset\}$. (It is a subscheme of S.)
Assume that the *very general* point $o \in S$ has the following properties:
(a) $Y_{os} = Y_o \cap Y_s$ is irreducible, for all $s \in S(o)$, and the intersection is transverse for $s \in S(o)$ generic;
(b) For $s \in S(o)$ generic, all the arrows below are isomorphisms:

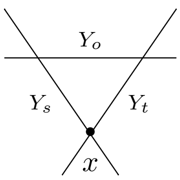
$$\begin{array}{ccccc} & \xrightarrow{\operatorname{res}^X_{Y_s}} & \operatorname{Pic}(Y_s) & \xrightarrow{\operatorname{res}^{Y_s}_{Y_{os}}} & \\ \operatorname{Pic}(X) & & \xrightarrow{\operatorname{res}^X_{Y_{os}}} & & \operatorname{Pic}(Y_{os}); \\ & \xrightarrow{\operatorname{res}^X_{Y_o}} & \operatorname{Pic}(Y_o) & \xrightarrow{\operatorname{res}^{Y_o}_{Y_{os}}} & \end{array} \qquad \text{(Pic)}$$

(c) $\rho(\mathcal{Y}_{S(o)}) \subset X$ is open: (YSX)



The (1-arm) condition: these configurations cover an open neighbourhood of Y_o. (1-arm)

(d) For all $s, t \in S(o)$ holds:
$Y_{st} \neq \emptyset \;\Rightarrow\; Y_{ost} \neq \emptyset$ and connected.



The (no-△) condition: such configurations should *not* exist. (no-△)

Remark 2.2 If all the triple intersections are non-empty and connected (sufficient conditions are given in 1.15), then $S(o) = S$, and (1-arm), (no-△) are automatically satisfied. This property holds in the case of zero loci of sections in ample vector bundles, which was studied in [12]. More general situations when (YSX) is satisfied appear in 3.6, 4.8.

However, for the isotropic symplectic and orthogonal Grassmannian studied in section 6, the generic double and triple intersections are empty. The analysis of these cases led to the weaker conditions above.

Definition 2.3 (i) The *geometric generic* fibre of π is defined by the Cartesian diagram

$$\begin{array}{ccc} \mathbb{Y} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow{\scriptstyle\pi} \\ \mathrm{Spec}(\bar{\Bbbk}) & \longrightarrow & S, \end{array} \qquad \Bbbk := \mathbb{C}(S).$$

A *very general point* of S refers to a point outside a countable union of subvarieties of S.
(ii) The *double* (respectively *triple*) *geometric generic self-intersections* of $\mathcal{Y}$, denoted $\mathbb{Y}_2$ and $\mathbb{Y}_3$ respectively, are defined by means of the diagrams:

$$\begin{array}{ccccc} \mathcal{Y}_2 := \overline{(\mathcal{Y}\times_X\mathcal{Y})\setminus\mathrm{diag}(\mathcal{Y})} & \hookrightarrow & \mathcal{Y}\times_X\mathcal{Y} & \longrightarrow & X \\ \Big\downarrow{\scriptstyle\pi_2} & & \downarrow & & \downarrow{\scriptstyle\mathrm{diag}} \\ & & \mathcal{Y}\times\mathcal{Y} & \longrightarrow & X\times X \\ & & \downarrow & & \\ S_2 := \mathrm{Supp}\,(\pi\times\pi)_*\mathcal{O}_{\mathcal{Y}_2} & \hookrightarrow & S\times S & & \end{array} \qquad \begin{array}{c} \mathcal{Y}_3 \hookrightarrow \mathcal{Y}\times_X\mathcal{Y}\times_X\mathcal{Y}\to X \\ \downarrow \\ S\times S\times S. \end{array} \tag{2.3}$$

$$\mathbb{Y}_2 := \mathcal{Y}_2\times\mathrm{Spec}\left(\overline{\mathbb{C}(S_2)}\right), \qquad \mathbb{Y}_3 \text{ is defined similarly.}$$

Thus there are natural morphisms $\mathbb{Y}_2 \rightrightarrows \mathbb{Y}$ (the projections onto the first and second components), and $\mathbb{Y}_3 \overset{\to}{\rightrightarrows} \mathbb{Y}_2$. (One may think off $\mathbb{Y}_2$ as $Y_{st} = Y_s\cap Y_t$, for very general $(s,t)\in S_2$, and the morphisms are the inclusions into Y_s, Y_t respectively; similarly for $\mathbb{Y}_3$.)

For $o\in S$, $S(o)$ introduced at (YSX) is $(\mathrm{pr}_S^{S_2})^{-1}(o)\subset(\{o\}\times S)\cap S_2$.

Lemma 2.4 *There is an open analytic subset (a ball) $\mathbb{B}\subset S$, which can be chosen arbitrarily in some Zariski open subset of S, such that the following hold:*
(i) If $(\rho^*\mathscr{V})\otimes\bar{\Bbbk}$ splits on $\mathbb{Y}$, then $(\rho^*\mathscr{V})_{\mathbb{B}}$ splits.
(ii) If $\mathrm{Pic}(X\otimes_{\mathbb{C}}\bar{\Bbbk})\to\mathrm{Pic}(\mathbb{Y})$ *and* $\mathrm{Pic}(\mathbb{Y})\rightrightarrows\mathrm{Pic}(\mathbb{Y}_2)$ *are isomorphisms, then* (YSX)(Pic) *is satisfied for points in $\mathbb{B}$.*
This condition is satisfied in the particular case when $\mathcal{Y}\subset S\times X$ is a δ-codimensional, $p^{>0}$ family of subvarieties of X as in 1.11, with $p\geqslant\delta+3$.

Proof. Since $(\rho^*\mathscr{V})_{\mathbb{Y}}$ splits, there are $\ell'_1,\dots,\ell'_r\in\mathrm{Pic}(\mathbb{Y})$ such that $(\rho^*\mathscr{V})_{\mathbb{Y}}=\ell'_1\oplus\ldots\oplus\ell'_r$; the right hand side is defined over a finitely generated (algebraic) extension of $\mathbb{C}(S)$. Thus there is an open affine $S^\circ\subset S$, and a finite morphism $\sigma: S'\to S^\circ$ such that $\ell'_1,\dots,\ell'_r$ are defined over $\mathbb{C}[S']$, and $(\rho^*\mathscr{V})_{S'}$ splits on $\mathcal{Y}_{S'}$. Then there are open balls $\mathbb{B}'\subset S'$ and $\mathbb{B}\subset S^\circ$ such that $\mathbb{B}'\overset{\sigma}{\to}\mathbb{B}$ is an *analytic* isomorphism, and the splitting of $(\rho^*\mathscr{V})_{\mathbb{B}'}$ on $\mathcal{Y}_{\mathbb{B}'}$ descends to $(\rho^*\mathscr{V})_{\mathbb{B}}$ on $\mathcal{Y}_{\mathbb{B}}$. The second statement is analogous.

In the situation 1.11, $\mathrm{Pic}(X_{\bar{\Bbbk}})\to\mathrm{Pic}(\mathbb{Y})$ is an isomorphism for $p\geqslant 3$, cf. theorem 1.18; the isomorphism $\mathrm{Pic}(\mathbb{Y})\to\mathrm{Pic}(\mathbb{Y}_2)$, requires $p-\delta\geqslant 3$. □

Henceforth, for $o\in S$ very general, we replace S by $S(o)$ (actually by an irreducible component of it containing o) and $\mathcal{Y}$ by $\mathcal{Y}\times_S S(o)$; the restrictions of π,ρ are denoted the same.

Lemma 2.5 $\mathrm{Pic}(X)\overset{\rho^*}{\to}\mathrm{Pic}(\mathcal{Y}_S)$ *is an isomorphism, for S small enough.*

Proof. The composition $\mathrm{Pic}(X)\overset{\rho^*}{\to}\mathrm{Pic}(\mathcal{Y})\overset{\mathrm{res}_{Y_o}}{\to}\mathrm{Pic}(Y_o)$ is bijective, so ρ^* is injective. For the surjectivity, take $\ell\in\mathrm{Pic}(\mathcal{Y})$. If $\ell_{Y_o}\cong\mathcal{O}_{Y_o}$, then

$$\{s\in S\mid \ell_{Y_s}\ncong\mathcal{O}_{Y_s}\}=\{s\in S\mid h^0(\ell_{Y_s})=0\}$$

is open, by semi-continuity, so $\{s \in S \mid \ell_{Y_s} \cong \mathcal{O}_{Y_s}\}$ is closed. On the other hand, by restricting to Y_{os}, the hypothesis (Pic) implies that this set is dense; thus it is the whole S. It follows that $\ell \cong \pi^*\bar{\ell}$, with $\bar{\ell} \in \mathrm{Pic}(S)$. But $\mathrm{Pic}(S)$ is trivial for S sufficiently small, so $\ell \cong \mathcal{O}$. If $\ell \in \mathrm{Pic}(\mathcal{Y})$ is arbitrary, take $\mathcal{L} \in \mathrm{Pic}(X)$ such that $\ell_{Y_o} \cong \mathcal{L}_{Y_o}$, so $(\rho^*\mathcal{L}^{-1})\ell|_{Y_o}$ is trivial. □

Let $\mathscr{V}$ be a vector bundle on X, such that $\rho^*\mathscr{V}$ splits; by the previous lemma,

$$\rho^*\mathscr{V} \cong \rho^*\big(\underset{j\in J}{\oplus}\mathcal{L}_j^{\oplus m_j}\big), \text{ with } \mathcal{L}_j \in \mathrm{Pic}(X) \text{ pairwise non-isomorphic.} \tag{2.4}$$

For any $s \in S$, let $M_s \subset J$ be the subset of maximal elements, for $\mathscr{V}_{Y_s}$. By semi-continuity, there is a neighbourhood $S_s \subset S$ of s such that $M_s \subset M_{s'}$ for all $s' \in S_s$; thus there is a largest subset $M \subset J$, and an open subset $S' \subset S$ such that $M = M_s$, for all $s \in S'$. Hence, possibly after shrinking S, the set of maximal isotypical components of $\mathscr{V}_{Y_s}$ with respect to (2.1) is independent of $s \in S$.

Lemma 2.6 *Let the situation be as above, and $\mathbb{B} \subset S$ be a standard (analytic) ball. We consider the (analytic) open subset $\mathcal{U} := \rho(\mathcal{Y}_{\mathbb{B}}) \subset X$. Then the following statements hold:*

(i) *There is a pointwise injective homomorphism $\big(\bigoplus_{\mu\in M}\mathcal{L}_\mu^{\oplus m_\mu}\big)\otimes\mathcal{O}_{\mathcal{U}} \to \mathscr{V}\otimes\mathcal{O}_{\mathcal{U}}$ whose restriction to Y_s is the natural evaluation* (2.2)*, for all $s \in \mathbb{B}$.*

(ii) *$\mathscr{V}\otimes\mathcal{O}_{\mathcal{U}}$ is obtained as a successive extension of $\{\mathcal{L}_j\}_{j\in J} \subset \mathrm{Pic}(X)$.*

Proof. (i) First we prove that $\mathcal{U}$ is indeed open. Since ρ is an algebraic morphism, $\rho(\mathcal{Y}_{\mathbb{B}}) \subset X$ is a locally closed analytic subvariety (it is constructible). If it is not open, the (local) components of $\rho|_{\mathcal{Y}_{\mathbb{B}}}$ satisfy a non-trivial algebraic relation. This relation holds on whole $\mathcal{Y}$, since $\mathcal{Y}_{\mathbb{B}} \subset \mathcal{Y}$ is open, which contradicts the hypothesis that $\rho(\mathcal{Y}) \subset X$ is open.

Now we proceed with the proof of the lemma. For all $s \in \mathbb{B}$, the restriction of

$$\mathrm{ev} : \bigoplus_{\mu\in M}\rho^*\mathcal{L}_\mu \otimes \pi^*\pi_*\rho^*(\mathcal{L}_\mu^{-1}\otimes\mathscr{V})_{\mathbb{B}} \to (\rho^*\mathscr{V})_{\mathbb{B}}$$

to Y_s is the homomorphism (2.2). The maximality of $\mu \in M$ implies that

$$\pi_*\rho^*(\mathcal{L}_\mu^{-1}\otimes\mathscr{V}) \cong \mathcal{O}_{\mathbb{B}}^{\oplus m_\mu}, \forall\, \mu \in M,$$

and ev is pointwise injective. We claim that, after suitable choices of bases in $\pi_*\rho^*(\mathcal{L}_\mu^{-1}\otimes\mathscr{V})$, the homomorphism ev descends to $\mathcal{U}$. We deal with each $\mu \in M$ separately, the overall basis being the direct sum of the individual ones.

Consider $\mu \in M$, and a base point $o \in \mathbb{B}$. Then $\mathscr{V}' := \mathcal{L}_\mu^{-1}\otimes\mathscr{V}$ has the properties:

– $(\rho^*\mathscr{V}')_{\mathbb{B}} \cong \mathcal{O}_{\mathcal{Y}_{\mathbb{B}}}^{\oplus m} \oplus \bigoplus_{j\in J\setminus\{\mu\}}\rho^*(\mathcal{L}_\mu^{-1}\mathcal{L}_j)_{\mathbb{B}}^{\oplus m_j}$.

– $\pi_*(\rho^*\mathscr{V}')_{\mathbb{B}} \cong \mathcal{O}_{\mathbb{B}}^{\oplus m}$. We choose an isomorphism $\alpha_{\mathbb{B}}$ between them.

– $\pi^*\pi_*(\rho^*\mathscr{V}')_{\mathbb{B}} \to (\rho^*\mathscr{V}')_{\mathbb{B}}$ is pointwise injective; let $\mathcal{T} \subset (\rho^*\mathscr{V}')_{\mathbb{B}}$ be its image.

We choose a complement $\mathcal{W} \cong \bigoplus_{j\in J\setminus\{\mu\}}\rho^*(\mathcal{L}_\mu^{-1}\mathcal{L}_j)^{\oplus m_j}$ of $\mathcal{T}$, so $(\rho^*\mathscr{V}')_{\mathbb{B}} = \mathcal{T}\oplus\mathcal{W}$. ($\mathcal{W}$ is defined up to $\mathrm{Hom}(\rho^*\mathscr{V}/\mathcal{T},\mathcal{T})$.) Then $\alpha_{\mathbb{B}}$ above determines the pointwise injective homomorphism $\alpha : \mathcal{O}_{\mathcal{Y}_{\mathbb{B}}}^{\oplus m} \to (\rho^*\mathscr{V}')_{\mathbb{B}} = \mathcal{T}\oplus\mathcal{W}$ whose second component vanishes, since $\Gamma(\mathcal{Y}_{\mathbb{B}},\mathcal{W}) = 0$. The left inverse $\beta : (\rho^*\mathscr{V}')_{\mathbb{B}} \to \mathcal{O}_{\mathcal{Y}_{\mathbb{B}}}^{\oplus m}$ of α with respect to the splitting, satisfies $\alpha\circ\beta|_{\mathcal{T}} = \mathbb{1}_{\mathcal{T}}$.

Claim After a suitable change of coordinates in $\mathcal{O}_{\mathcal{Y}_{\mathbb{B}}}^{\oplus m}$, the homomorphisms α descends to $\rho(\mathcal{Y}_{\mathbb{B}}) \subset X$. Indeed, for any $s \in \mathbb{B}$, we consider the diagram (recall that $Y_{os} \neq \emptyset$):

$$\begin{array}{ccc} \mathcal{O}_{Y_{os}}^{\oplus m} & \xrightarrow{\alpha_o} & \mathscr{V}'_{Y_{os}} \\ {\scriptstyle a_s}\big\downarrow & & \big\| \\ \mathcal{O}_{Y_{os}}^{\oplus m} & \xrightarrow{\alpha_s} & \mathscr{V}'_{Y_{os}} \\ & \xleftarrow{\quad\beta_s\quad} & \end{array} \qquad \text{with } a_s := \beta_s \circ \alpha_o \in \mathrm{End}(\mathbb{C}^m).$$

Similarly, we let $a'_s := \beta_o \circ \alpha_s$. Then holds $a'_s a_s = \beta_o\alpha_s\beta_s\alpha_o = \beta_o\alpha_o = \mathbb{1}$ (the second equality uses $\mathrm{Im}\big(\alpha_o|_{Y_{os}}\big) = \mathcal{T}_{Y_{os}} = \mathrm{Im}\big(\alpha_s|_{Y_{os}}\big)$, and $\alpha_s\beta_s|_{\mathcal{T}} = \mathbb{1}$), and similarly $a_s a'_s = \mathbb{1}$. Thus $a_s \in \mathrm{GL}(m;\mathbb{C})$ for all $s \in \mathbb{B}$, and the new trivialization $\tilde{\alpha} := \alpha \circ a$ of $\mathcal{T}$ satisfies

$$\tilde{\alpha}_s = \tilde{\alpha}_o \text{ along } Y_{os}, \quad \forall\, s \in \mathbb{B}, \tag{2.5}$$

because $\tilde{\alpha}_s|_{Y_{os}} = (\alpha_s\beta_s)\alpha_o|_{Y_{os}} = \alpha_o|_{Y_{os}} = \tilde{\alpha}_o|_{Y_{os}}$.

Also, for all $s, t \in \mathbb{B}$ such that $Y_{st} \neq \emptyset$, the trivializations of $\mathcal{T}_{Y_{st}}$ induced by $\tilde{\alpha}$ from Y_s and Y_t, coincide; equivalently, the following diagram commutes:

$$\begin{array}{ccccc} \mathcal{O}_{Y_{st}}^{\oplus m} & \xrightarrow[\cong]{\tilde{\alpha}_s} & \mathcal{T}_{Y_{st}} & \subset & \mathscr{V}'_{Y_{st}} \\ \big\| & & \big\| & & \big\| \\ \mathcal{O}_{Y_{st}}^{\oplus m} & \xrightarrow[\cong]{\tilde{\alpha}_t} & \mathcal{T}_{Y_{st}} & \subset & \mathscr{V}'_{Y_{st}} \end{array} \qquad \Leftrightarrow\ \tilde{\alpha}_t^{-1} \circ \tilde{\alpha}_s|_{Y_{st}} = \mathbb{1} \in \mathrm{GL}(r;\mathbb{C}). \tag{2.6}$$

Indeed, Y_{ost} is non-empty and connected by (no-$\triangle$), so is enough to prove that the restriction of (2.6) to Y_{ost} is the identity. After restricting (2.5) to Y_{ost}, we deduce

$$\tilde{\alpha}_s|_{Y_{ost}} = \tilde{\alpha}_o|_{Y_{ost}} = \tilde{\alpha}_t|_{Y_{ost}} \quad \Rightarrow \quad \tilde{\alpha}_t^{-1} \circ \tilde{\alpha}_s|_{Y_{ost}} = \mathbb{1}.$$

Now we can conclude that the trivialization $\tilde{\alpha}$ of $\pi_*\rho^*(\mathcal{L}_\mu^{-1} \otimes \mathscr{V})$ descends to $\mathcal{U} := \rho(\mathcal{Y}_{\mathbb{B}})$, as announced. Indeed, define

$$\bar{\alpha} : \mathcal{O}_{\mathcal{U}}^{\oplus m} \to \mathscr{V} \otimes \mathcal{O}_{\mathcal{U}},\ \bar{\alpha}(x) := \tilde{\alpha}_s(x) \text{ for some } s \in \mathbb{B} \text{ such that } x \in Y_s.$$

The diagram (2.6) implies that $\bar{\alpha}(x)$ is independent of $s \in \mathbb{B}$ with $s(x) = 0$.
(ii) Apply repeatedly the first part. □

Lemma 2.7 *Let $Y_o \subset X$ be a subvariety and $\mathcal{U} \supset Y_o$ be an open neighbourhood of it. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be vector bundles on X, whose restriction to $\mathcal{U}$ fit into $0 \to \mathcal{A}_{\mathcal{U}} \to \mathcal{B}_{\mathcal{U}} \to \mathcal{C}_{\mathcal{U}} \to 0$. Assume that either one of the following conditions is satisfied:*
(i) *$Y_o \subset X$ is $2^{\gtrsim 0}$ (cf. 1.7);*
or (ii) *$\mathrm{cd}(X \setminus Y_o) \leqslant \dim X - 3$.*
Then $\mathcal{B}$ is an extension of $\mathcal{C}$ by $\mathcal{A}$ on X.

Proof. (i) As Y_o is $2^{\gtrsim 0}$, $\mathrm{Ext}^1(\mathcal{C}, \mathcal{A}) \to \mathrm{Ext}^1(\mathcal{C}_{\tilde{Y}_m}, \mathcal{A}_{\tilde{Y}_m})$ is an isomorphism for an increasing sequence of thickenings $\{\tilde{Y}_m\}_{m \geqslant m_0}$ of Y_o. The restriction of $\mathcal{B}_{\mathcal{U}} \in \mathrm{Ext}^1(\mathcal{C}_{\mathcal{U}}, \mathcal{A}_{\mathcal{U}})$ to $\tilde{Y}_m$ yields the extension $0 \to \mathcal{A} \to \mathcal{B}' \to \mathcal{C} \to 0$ over X, with the property that $\mathcal{B}'_{\tilde{Y}_m} \cong \mathcal{B}_{\tilde{Y}_m}$, for all $m \geqslant m_0$. As Y_o is $2^{\gtrsim 0}$, it follows $\mathcal{B} \cong \mathcal{B}'$.
(ii) The hypothesis that $\mathrm{cd}(X \setminus Y_o) \leqslant \dim X - 3$ implies (cf. [15, Ch. III, Theorem 3.4]) that for any locally free sheaf $\mathcal{F}$ on X the restriction below is an isomorphism:

$$H^i(X, \mathcal{F}) \to H^i(\hat{X}_{Y_o}, \hat{\mathcal{F}}),\ \text{for } i = 0, 1.$$

Now use that $\mathrm{Ext}^1\big(\mathcal{C}_{\hat{X}_{Y_o}},\mathcal{A}_{\hat{X}_{Y_o}}\big)\cong H^1\big(\mathcal{H}om(\mathcal{C}_{\hat{X}_{Y_o}},\mathcal{A}_{\hat{X}_{Y_o}})\big)$ and proceed as before. □

2.2. **The splitting criterion.** Let $\mathbb{F}\hookrightarrow\mathbb{C}$ be a finitely generated extension of $\mathbb{Q}$, such that $X,\mathcal{Y},\mathscr{V}$ are defined over $\mathbb{F}$; its algebraic closure $\bar{\mathbb{F}}\subset\mathbb{C}$ is countable.

Theorem 2.8 *Let X be a a smooth, projective variety, and assume the following:*

(i) *the situation is as in* (YSX)*;*
(ii) $\mathscr{V}_{\mathbb{Y}}$ *splits on* $\mathbb{Y}$*; alternatively,* $\mathscr{V}_{Y_s}$ *splits, for a very general* $s\in S$*;*
(iii) $\mathbb{Y}\subset X\otimes_{\mathbb{C}}\bar{\Bbbk}=:X_{\bar{\Bbbk}}$ *is either* $2^{\gtrsim 0}$ *(e.g. it is* $2^{>0}$*) or* $\mathrm{cd}\big(X_{\bar{\Bbbk}}\setminus\mathbb{Y}\big)\leqslant\dim X-3$.

Then $\mathscr{V}$ is obtained by successive extensions of line bundles on X. If, moreover, X has the property that $H^1(X,\mathcal{L})=0$ for all $\mathcal{L}\in\mathrm{Pic}(X)$, then $\mathscr{V}$ splits.

The very same statements remain valid if X is defined over an uncountable, algebraically closed field, rather than over $\mathbb{C}$.

Proof. First assume that $\mathscr{V}_{\mathbb{Y}}$ splits. By lemma 2.4, there is a ball $\mathbb{B}\subset S$, such that $(\rho^*\mathscr{V})_{\mathcal{Y}_{\mathbb{B}}}$ splits; lemma 2.6 implies that $\mathscr{V}$ is a successive extension of line bundles on a tubular neighbourhood of Y_o, $o\in\mathbb{B}$. It remains to apply lemma 2.7.

Let $\tau:S\to S_{\bar{\mathbb{F}}}$ be the trace morphism; for $s\in S$, let $\Bbbk_0:=\bar{\mathbb{F}}\big(\tau(s)\big)$ be the residue field of $\tau(s)\in S_{\bar{\mathbb{F}}}$. For s very general, $\tau(s)$ is the generic point of $S_{\bar{\mathbb{F}}}$, so $\Bbbk_0=\bar{\mathbb{F}}\big(S_{\bar{\mathbb{F}}}\big)$. Assume $\mathscr{V}_s$ splits; in the Cartesian diagram below $\mathscr{V}_s=\mathscr{V}_{\bar{\Bbbk}_0}\otimes\mathbb{C}$:

$$\begin{array}{ccccc} Y_s & \longrightarrow & \mathbb{Y}_{\bar{\Bbbk}_0} & \longrightarrow & \mathcal{Y}_{\bar{\mathbb{F}}}\\ \downarrow & & \downarrow & & \downarrow\\ \mathrm{Spec}(\mathbb{C}) & \longrightarrow & \mathrm{Spec}(\bar{\Bbbk}_0) & \longrightarrow & S_{\bar{\mathbb{F}}}. \end{array}$$

The splitting of a vector bundle commutes with base change, for varieties defined over algebraically closed fields (cf. [12]). The previous discussion implies that $\mathscr{V}_{\bar{\Bbbk}_0}$ splits on $\mathbb{Y}_{\bar{\Bbbk}_0}$ (the geometric generic fibre of $\mathcal{Y}_{\bar{\mathbb{F}}}\to S_{\bar{\mathbb{F}}}$); hence the same holds for $\mathscr{V}_{\mathbb{Y}}$ on $\mathbb{Y}=\mathbb{Y}_{\bar{\Bbbk}_0}\otimes\mathbb{C}$. This brings us back to the previous case. For the final statement, we use once more that the splitting property commutes with base change. □

The proof of the theorem even precises the meaning of the term 'very general': if $\mathbb{F}$ is the field of definition of $X,\mathcal{Y},\mathscr{V}$, then, in local affine coordinates coming from $S_{\mathbb{F}}$, the coordinates of $s\in S$ should be algebraically independent over $\mathbb{F}$.

Definition 2.9 We say that the variety X is 1-*splitting* if $H^1(X,\mathcal{L})=0$, for all $\mathcal{L}\in Pic(X)$.

The simplest examples of 1-splitting varieties are the Fano varieties of dimension at least two with cyclic Picard group, and products of such. In 5.5 are obtained examples of homogeneous 1-splitting varieties with Picard groups of higher rank.

Remark 2.10 The genericity assumption in 2.8 plays an *essential* role. If one is interested in the same result for arbitrary Y, one needs stronger positivity hypotheses, in order to apply effective cohomology vanishing results; in [12] I obtained a similar result for *ample*, globally generated vector bundles. In this case, the situation (YSX) holds automatically.

Theorem 2.8 will be illustrated in the following sections with concrete examples.

3. POSITIVITY PROPERTIES OF ZERO LOCI OF SECTIONS IN VECTOR BUNDLES

Throughout this section, $\mathcal{N}$ stands for a *globally generated* vector bundle on X of rank ν.

3.1. **Sommese's q-ampleness for globally generated vector bundles.** We briefly review the q-ampleness concept introduced in [24].

Proposition 3.1 (cf. [24, Proposition 1.7]). *The following statements are equivalent:*

(i) *The vector bundle $\mathcal{N}$ is q-ample (cf. definition 1.1), that is for all coherent sheaves $\mathcal{F}$ on X holds*
$$H^t(X, \mathcal{F} \otimes \mathrm{Sym}^m(\mathcal{N})) = 0,\ \forall\, t \geqslant q+1,\ \forall\, m \gg 0; \tag{3.1}$$

(ii) *The fibres of the morphism $\mathbb{P}(\mathcal{N}^\vee) \to |\mathcal{O}_{\mathbb{P}(\mathcal{N}^\vee)}(1)|$ are at most q-dimensional.*

Proposition 3.2 *Assume $\mathcal{N}$ is q-ample. Then $\mathcal{O}_{\mathbb{P}(\mathcal{N}^\vee)}(1)$ is q-positive (as in 1.12). If Y is the zero locus of a* regular *section in $\mathcal{N}$, then $Y \subset X$ is* $(\dim X - \nu - q)^{>0}$.

Proof. The first statement is proved in [20, Theorem 1.4]. Since the section is regular, $\mathrm{codim}_X(Y) = \nu$. For any coherent sheaf $\mathcal{F}$ on X holds
$$H^{\nu+t}(\mathbb{P}(\mathcal{N}), \mathcal{O}(-\nu-m) \otimes \pi^*(\mathcal{F} \otimes \det(\mathcal{N})^{-1})) = H^{t+1}(X, \mathcal{F} \otimes \mathrm{Sym}^m(\mathcal{N})),$$
so the $H^{\geqslant \nu+q}$-cohomology on $\mathbb{P}(\mathcal{N})$ vanishes; hence the same holds for $\tilde{X} \subset \mathbb{P}(\mathcal{N})$. □

For $\nu = 1$, the line bundle $\mathcal{N}$ is q-ample if an only if the morphism $X \to |\mathcal{N}|$ has at most q-dimensional fibres. This property is easy to check, and convenient for concrete applications. In contrast, for $\nu \geqslant 2$, the criterion is not effective; the q-ampleness test for $\mathcal{N}$ is too restrictive to check the positivity of the zero loci of its sections (cf. remarks 3.4, 6.5.)

3.2. **The positivity criterion 1.10.** Suppose $Y \subset X$ is lci of codimension δ, and the zero locus of a section s in $\mathcal{N}$, of rank $\nu \geqslant 2$; we *do not assume* that s is regular, so we allow $\delta < \nu$. In this context, the situation 1.10 arises as follows: since Y is the zero locus of $s \in \Gamma(\mathcal{N})$, the blow-up $\tilde{X}$ fits into

$$\tilde{X} \hookrightarrow \mathbb{P}(\mathcal{N}) = \mathbb{P}\Big(\bigwedge^{\nu-1} \mathcal{N}^\vee \otimes \det(\mathcal{N})\Big) \hookrightarrow X \times \mathbb{P}\Big(\bigwedge^{\nu-1} \Gamma(\mathcal{N})^\vee\Big) \tag{3.2}$$
$$\pi: \tilde{X} \to X, \qquad b: \tilde{X} \to \mathbb{P} := \mathbb{P}\Big(\bigwedge^{\nu-1} \Gamma(\mathcal{N})^\vee\Big),$$

and holds
$$\mathcal{O}_{\tilde{X}}(E_Y) = \mathcal{O}_{\mathbb{P}(\mathcal{N})}(-1)\big|_{\tilde{X}} = \big(\det(\mathcal{N}) \boxtimes \mathcal{O}_{\mathbb{P}}(-1)\big)\big|_{\tilde{X}}. \tag{3.3}$$

Proposition 3.3 *Suppose* $\det(\mathcal{N})$ *is ample. If the dimension of the generic fibre of b (over its image) is $p+1$, then $\mathcal{O}_{\tilde{X}}(E_Y)$ is $\dim b(\tilde{X})$-positive, and Y is $p^{>0}$.*

Proof. The assumptions of the proposition 1.10 are satisfied. □

Observe that the propositions 3.1 and 3.3 deal with complementary situations: $\mathcal{O}_{\mathbb{P}(\mathcal{N}^\vee)}(1)$ is the pull-back of an ample line bundle; $\mathcal{O}_{\tilde{X}}(E_Y)$ is relatively ample for some morphism.

Let $W \subseteq \Gamma(\mathcal{N})$ be a vector subspace which generates $\mathcal{N}$; let $\dim W = \nu + u + 1$. A globally generated vector bundle $\mathcal{N}$ on X is equivalent to a morphism $f : X \to \mathrm{Gr}(W; \nu)$ into the Grassmannian of ν-dimensional quotients of W; $\det(\mathcal{N})$ is ample if and only if φ is finite onto its image.

For $\mathrm{Gr}(W;\nu)$ and $\mathcal{N}$ the universal quotient bundle on it, we can explicitly write the morphism b in (3.2): $\mathbb{P}(\mathcal{N}) \to \mathbb{P}$ is defined by

$$(x, \langle e_x\rangle) \mapsto \det(\mathcal{N}_x/\langle e_x\rangle)^\vee \subset \bigwedge^{\nu-1}\mathcal{N}_x^\vee \subset \bigwedge^{\nu-1} W^\vee. \tag{3.4}$$

($\langle e_x\rangle$ stands for the line generated by $e_x \in \mathcal{N}_x$, $x \in \mathrm{Gr}(W;\nu)$.) The restriction to the Grassmannian corresponds to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathrm{Gr}(W;\nu)} & \xrightarrow{s} & W\otimes\mathcal{O}_{\mathrm{Gr}(W;\nu)} & \longrightarrow & W/\langle s\rangle\otimes\mathcal{O}_{\mathrm{Gr}(W;\nu)} & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow & & \\ & & \mathcal{O}_{\mathrm{Gr}(W;\nu)} & \xrightarrow{\beta s} & \mathcal{N} & \longrightarrow & \mathcal{N}/\langle\beta s\rangle & \longrightarrow & 0. \end{array} \tag{3.5}$$

Thus b is the desingularization of the rational map

$$g_s : \mathrm{Gr}(W;\nu) \dashrightarrow \mathrm{Gr}(W/\langle s\rangle;\nu-1), \quad [W \twoheadrightarrow N] \mapsto [W/\langle s\rangle \twoheadrightarrow N/\langle\beta s\rangle] \tag{3.6}$$

followed by the Plücker embedding of $\mathrm{Gr}(W/\langle s\rangle;\nu-1)$; the indeterminacy locus of b is $\mathrm{Gr}(W/\langle s\rangle;\nu) \subset \mathrm{Gr}(W;\nu)$.

Remark 3.4 We mentioned that Sommese's q-ampleness criterion is not effective for $\nu \geqslant 2$. For $X = \mathrm{Gr}(\nu+u+1;\nu)$, Sommese's criterion implies that $\mathcal{N}$ is q-ample, with $q = \dim\mathbb{P}(\mathcal{N}^\vee) - \mathbb{P}^{\nu+u} = \dim X - (u+1)$; hence $Y = \mathrm{Gr}(\nu+u;\nu)$, the zero locus of a generic section of $\mathcal{N}$, is $(u+1-\nu)^{>0}$. On the other hand, the criterion 1.10 implies that Y is actually $u^{>0}$.

There is a 'universal' rational map g_{univ} containing the maps g_s above, as s varies:

$$\begin{aligned} g_{\mathrm{univ}} : \mathbb{P}(W)\times\mathrm{Gr}(W;\nu) &\dashrightarrow \mathrm{Flag}(W;\nu+u,\nu-1), \\ (\langle s\rangle, [W\twoheadrightarrow N]) &\mapsto \big[W \twoheadrightarrow \tfrac{W}{\langle s\rangle} \twoheadrightarrow \tfrac{N}{\langle\beta(s)\rangle}\big]. \end{aligned} \tag{3.7}$$

(The right hand side denotes the flag variety of successive quotients of W.) It is undefined on the incidence variety $\mathcal{J} := \{\, (\langle s\rangle, N) \mid s \in \mathrm{Ker}(W \twoheadrightarrow N)\}$.

Back to the general case of a globally generated vector bundle on a variety X. By varying $s \in W$, one obtains the family of subvarieties of X (over $\mathbb{P}(W)$)

$$\mathcal{Y} := (\mathbb{P}(W)\times X)\times_{\mathbb{P}(W)\times\mathrm{Gr}(W;\nu)}\mathcal{J},$$

and the situation mentioned at 1.11:

$$\begin{array}{ccccc} \mathrm{Bl}_{\mathcal{Y}}(\mathbb{P}(W)\times X) & \longrightarrow & \mathrm{Bl}_{\mathcal{J}}\big(\mathbb{P}(W)\times\mathrm{Gr}(W;\nu)\big) & \xrightarrow{\tilde g} & \\ \downarrow{\scriptstyle\pi} & \searrow{\scriptstyle b} & \downarrow & & \\ \mathbb{P}(W)\times X & \xrightarrow{\varphi} & \mathbb{P}(W)\times\mathrm{Gr}(W;\nu) & \overset{g_{\mathrm{univ}}}{\dashrightarrow} & \mathrm{Flag}(W;\nu+u;\nu-1). \end{array} \tag{3.8}$$

(The intersection $X\cap\mathrm{Gr}(W/\langle s\rangle;\nu)$ may not be transverse for certain $s \in W$.) Proposition 3.3 can be restated as follows.

Corollary 3.5 *Let $\varphi : X \to \mathrm{Gr}(W;\nu)$, $\nu \geqslant 2$, be a morphism finite onto its image. If the general fibre of $g_{\mathrm{univ}}\circ\varphi : \mathbb{P}(W)\times X \dashrightarrow \mathrm{Flag}(W;\nu+u;\nu-1)$ is at least $(p+1)$-dimensional, then Y_s is $p^{>0}$, for all $s \in W$ such that Y_s is irreducible lci in X.*

3.3. **The Picard group and the diagram** (YSX)**.** The sheaf $\mathcal{K}$ defined by

$$0 \to \mathcal{K} := \mathrm{Ker}(\mathrm{ev}) \to W \otimes \mathcal{O}_X \overset{\mathrm{ev}}{\to} \mathcal{N} \to 0$$

is locally free, and the incidence variety

$$\mathcal{Y} = \{(\langle s\rangle, x) \mid s(x) = 0\} \subset X \times \mathbb{P}(W) \tag{3.9}$$

is isomorphic to $\mathbb{P}(\mathcal{K})$. We denote π and ρ the projections onto $\mathbb{P}(W)$ and X respectively; actually we restrict ourselves to a sufficiently small open subset $S \subset \mathbb{P}(W)$. The vector bundles $\mathcal{N}^{\oplus 2}$ and $\mathcal{N}^{\oplus 3}$ are generated by $W^{\oplus 2}$ and $W^{\oplus 3}$ respectively, and determine the double and triple self-intersection diagrams (2.3).

Proposition 3.6 *Let $\mathcal{Y} \subset S \times X$ be as in* (3.9)*, with $S \subset \mathbb{P}(W)$ suitably small. Then the conditions of* (YSX) *are fulfilled as soon as $\mathcal{N}$ satisfies any of the following conditions:*

(i) *$\mathcal{N}$ is Sommese-q-ample (cf. 3.1), and*
$\dim X - q \geqslant 3\nu + 1$ *(for $\nu \geqslant 2$), or* $\dim X - q \geqslant 5$ *(for $\nu = 1$).*

or

(ii) *$\nu \geqslant 2$, and the generic fibre of $\mathbb{P}(W) \times X \dashrightarrow \mathrm{Flag}(W; \nu + u; \nu - 1)$ is at least $2(\nu + 1)$-dimensional (cf.* (3.8)*).*

Proof. (i) The non-emptiness of the triple intersections requires $\dim X - q \geqslant 3\nu + 1$, by lemma 1.15; the isomorphism of Picard groups requires $\dim X - q \geqslant 2\nu + 3$, by theorem 1.18.
(ii) In this case, $Y_s \subset X$ is $(2\nu + 1)^{>0}$, so the triple intersections are non-empty (cf. 1.15); the isomorphism of Picard groups requires $(2\nu + 1) - \nu \geqslant 3$ (cf. 2.4). □

3.4. **Splitting along zero loci of globally generated vector bundles.** Let $\mathscr{V}$ be an arbitrary vector bundle on X. The previous discussion immediately yields the following criterion.

Theorem 3.7 *Let X be 1-splitting variety (cf. 2.9), $\mathcal{N}$ be a globally generated vector bundle of rank ν on X such that* $\det(\mathcal{N})$ *is ample, and $W \subset \Gamma(X, \mathcal{N})$ be a generating vector subspace. Assume that $\mathcal{N}$ satisfies one of the conditions in proposition 3.6.*

Then $\mathscr{V}$ splits on X, if and only if its restriction to the zero locus of a very general $s \in W$ splits. (If X is not 1-splitting, $\mathscr{V}$ is a successive extension of line bundles.)

Example 3.8 Consider $\mathrm{Gr}(W; \nu)$, with $\nu \geqslant 2$, $\dim W = \nu + u + 1$, and let $X \subset \mathrm{Gr}(W; \nu)$ be an arbitrary subvariety (e.g. the Grassmannian itself) such that

$$\mathrm{Pic}(X) \cong \mathbb{Z}, \quad \mathrm{codim}_{\mathrm{Gr}(W;\nu)}(X) \leqslant u - (2\nu + 1).$$

Then a vector bundle $\mathscr{V}$ splits if and only if it does so along a very general zero locus $Y \subset X$ of a section in $\mathcal{N}_X$. Indeed, in this case proposition 3.6(ii) is satisfied. Moreover, observe that $\mathrm{Pic}(Y) \cong \mathrm{Pic}(X) \cong \mathbb{Z}$ so the procedure can be iterated as long as the codimension condition above is satisfied.

In section 6, this principle will be further illustrated with numerous concrete examples.

Is natural to ask what happens if one drops the hypothesis that $\det(\mathcal{N})$ is ample. The Stein factorization of $\varphi : X \to \mathrm{Gr}(W; \nu)$ decomposes it into a morphism with connected fibres followed by a finite map.

Theorem 3.9 *Let $\varphi : X \to X'$ be a smooth morphism of relative dimension $d \geqslant 1$, $\mathcal{N}'$ a globally generated vector bundle on X' of rank ν with* $\det(\mathcal{N}')$ *ample, and $\mathcal{N} = \varphi^*\mathcal{N}'$.*

Assume that X is 1*-splitting, and $\mathcal{N}'$ satisfies proposition 3.6. Then $\mathscr{V}$ splits on X as soon as $\mathscr{V}$ splits along Y_s, for s very general.*

Proof. The conditions of theorem 2.8 are fulfilled. Indeed, consider the family $\mathcal{Y}'$ of subvarieties of X' as above and let $\mathcal{Y} := \varphi^{-1}(\mathcal{Y}')$. The positivity is preserved under pull-back (cf. lemma 1.9, 1.14). For the condition (Pic), the morphism $\tilde{\varphi} : \tilde{X} \to \tilde{X}'$ induced at the level of the blows-up is still smooth of relative dimension d, so $\mathcal{O}_{\tilde{X}}(E_Y) = \tilde{\varphi}^*\mathcal{O}_{\tilde{X}'}(E_{Y'})$ is $(q + d)$-ample; it remains to apply 1.18. □

It is surprising that this yields new results even in the simplest case $X = X' \times V$, with X', V smooth, and $\mathcal{N}' = \mathcal{O}_{X'}(1)$ is globally generated, ample. If X is 1-splitting, then X', V are both 1-splitting; if either X' or V are simply connected, the converse is true.

Corollary 3.10 *Assume that $X = X' \times V$ is* 1*-splitting,* $\dim X' \geqslant 5$*, and $\mathcal{O}_{X'}(1)$ is ample, globally generated. Then $\mathscr{V}$ splits on X, if it splits along a very general divisor in $|\varphi^*\mathcal{O}_{X'}(1)|$.*

4. Positivity properties of sources of G_m-actions

Another (totally different) framework which leads to the situation (YSX) arises in the context of the actions of the multiplicative group on (almost) homogeneous varieties.

4.1. **Basic properties of the BB-decomposition.**

We start with general considerations which should justify the appearance of the homogeneous varieties in the next section. Let G be a connected reductive linear algebraic group, and $T \subset B \subset G$ be a maximal torus and a Borel subgroup. Finally, let X be a smooth projective G-variety with a faithful G-action $\mu : G \times X \to G \times X$.

Now consider a 1-parameter subgroup (1-PS for short) $\lambda : G_m \to G$, where $G_m = \mathbb{C}^*$ is the multiplicative group; we assume $\lambda(G_m) \subset T$. The 1-PS induces the action $\lambda : G_m \times X \to X$, which determines the so-called Bialynicki-Birula (BB for short) decomposition of X into locally closed subsets. Below are summarized its basic properties (cf. [6, 17]):

- The specializations at $\{0, \infty\} = \mathbb{P}^1 \setminus G_m$ are denoted $\lim_{t\to 0}\lambda(t) \times x$ and $\lim_{t\to\infty}\lambda(t) \times x$; they are both fixed by λ.
- The fixed locus X^λ of the action is a disjoint union $\coprod_{s\in S_{\mathrm{BB}}} Y_s$ of smooth subvarieties. For $s \in S_{\mathrm{BB}}$, $Y_s^+ := \{x \in X \mid \lim_{t\to 0}\lambda(t) \times x \in Y_s\}$ is locally closed in X (a BB-cell).
- $X = \coprod_{s\in S_{\mathrm{BB}}} Y_s^+$, and the morphism $Y_s^+ \to Y_s,\ x \mapsto \lim_{t\to 0}\lambda(t) \times x$ is a locally trivial, affine space fibration; it is not necessarily a vector bundle.
- The *source* $Y := Y_{\mathrm{source}}$ and the *sink* Y_{sink} of the action are uniquely characterized by the fact that $Y^+ = Y^+_{\mathrm{source}} \subset X$ is open and $Y^+_{\mathrm{sink}} = Y_{\mathrm{sink}}$.
- By composing λ with the involution $t \mapsto t^{-1}$ of G_m, one gets the so-called *minus* BB-decomposition $X = \coprod_{s\in S_{\mathrm{BB}}} Y_s^-$.

We denote:

$$\begin{aligned} G(\lambda) &:= \big\{g\in G \mid g^{-1}\lambda(t)g=\lambda(t),\ \forall\, t\in G_m\big\} \text{ the centralizer of } \lambda \text{ in } G,\\ P(\pm\lambda) &:= \big\{g\in G \mid \lim_{t\to 0}\big(\lambda(t)^{\pm}g\,\lambda(t)^{\mp}\big) \text{ exists in } G\big\},\\ U(\pm\lambda) &:= \big\{g\in G \mid \lim_{t\to 0}\big(\lambda(t)^{\pm}g\,\lambda(t)^{\mp}\big)=e\in G\big\}. \end{aligned} \tag{4.1}$$

Then $G(\lambda)$ is a connected, reductive subgroup of G, $P(\pm\lambda)\subset G$ are parabolic subgroups, $G(\lambda)$ is their Levi-component, and $U(\pm\lambda)$ the unipotent radical (cf. [25, §13.4]).

Lemma 4.1 (i) *Y is invariant under $P(-\lambda)$, thus for $G(\lambda)$ too.*
(ii) *Y_s^+ is $P(\lambda)$-invariant and $U(\lambda)$ preserves the fibration $Y_s^+\to Y_s$, for all $s\in S_{\rm BB}$.*

Proof. See [13]. □

Theorem 2.8 requires that the embedded deformations of Y sweep out an open subset of X; this is achieved if GY, defined set theoretically as $\{gy \mid g\in G,\ y\in Y\}$, is open in X.

Lemma 4.2 *GY has a natural structure of a closed subscheme of X. Therefore $GY\subset X$ is open if and only if $GY=X$.*

Proof. Indeed GY is the image of $\mu: G\times Y\to X$. Since Y is $P(-\lambda)$-invariant, it factorizes $(G\times Y)/P(-\lambda)\to X$, for $p\times(g,y):=(gp^{-1},py)$. But $P(-\lambda)$ is parabolic, so $(G\times Y)/P(-\lambda)$ is projective, hence $\mathrm{Im}(\mu)$ is a closed in X. □

Remark 4.3 Since $P(\lambda)P(-\lambda)Y=P(\lambda)Y=U(\lambda)G(\lambda)Y=U(\lambda)Y$, and $P(\lambda)P(-\lambda)$ is open in G, we deduce:

$$GY=X \;\Leftrightarrow\; U(\lambda)Y\subset X \text{ is open.}$$

This observation hints towards the fact that the G-varieties satisfying the lemma 4.2 should be homogeneous (or, at least, have an open B-orbit).

4.2. **The positivity of $Y\subset X$.** We follow the same steps as in the section 3: determine the positivity of the source of a G_m-action. In section 1 we obtained two methods for doing this.

4.2.1. *Apply the proposition 1.4.* Let us recall that $Y\subset X$ is $p^{>0}$ as soon as:
– $\mathcal{N}_{Y/X}$ is $(\dim X-p)$-ample;
– $\mathrm{cd}(X\setminus Y)\leqslant \dim X-(p+1)$.
It does not seem to exist a uniform answer for the amplitude of the normal bundle; in contrast, there is a compact formula for the cohomological dimension.

Proposition 4.4 *If $Y\subset X$ is the source of a G_m-action, then holds*

$$\mathrm{cd}(X\setminus Y)=\dim(X\setminus Y^+).$$

Observe that:
$$\begin{aligned} \dim X-\dim(X\setminus Y^+) &= \dim X-\max\{\dim \overline{Y_s^+} \mid s\neq \text{source}\}\\ &= \min_{s\neq\text{source}}\left\{\begin{matrix}\text{number of strictly negative}\\ \text{weights of } \lambda \text{ on } T_X|_{Y_s}\end{matrix}\right\}. \end{aligned} \tag{4.2}$$

Proof. See [13]. □

4.2.2. *Apply the proposition 1.10.* We remark that the G_m-action leads to the situation analyzed in proposition 1.10. By linearizing the G_m-action in a very ample line bundle on X, one gets a G_m-equivariant embedding of X into some $\mathbb{P}^N$, such that X is not contained in a hyperplane. In coordinates $z_{N_0} \in \mathbb{C}^{N_0}, \dots, z_{N_r} \in \mathbb{C}^{N_r}$, the G_m-action on $\mathbb{P}^N$ is:

$$t \times [z_{N_0}, z_{N_1}, \dots, z_{N_r}] = [z_{N_0}, t^{m_1} z_{N_1}, \dots, t^{m_r} z_{N_r}], \quad \text{with } 0 < m_1 < \dots < m_r. \tag{4.3}$$

The source and sink of $\mathbb{P}^N, X$ are respectively:

$$\begin{array}{ll} \mathbb{P}^N_{\text{source}} = \{[z_{N_0}, 0, \dots, 0]\}, & \mathbb{P}^N_{\text{sink}} = \{[0, \dots, 0, z_{N_r}]\}, \\ Y = Y_{\text{source}} = X \cap \mathbb{P}^N_{\text{source}}, & Y_{\text{sink}} = X \cap \mathbb{P}^N_{\text{sink}}, \\ Y^+ = X \cap (\mathbb{P}^N_{\text{source}})^+, & (\mathbb{P}^N_{\text{source}})^+ = \{\underline{z} = [z_{N_0}, z_{N_1}, \dots, z_{N_r}] \mid z_{N_0} \neq 0\}. \end{array} \tag{4.4}$$

(The intersections are set theoretical.) Note that $\mathbb{P}^N_{\text{source}}$ is the indeterminacy locus of the rational map

$$\mathbb{P}^N \dashrightarrow \mathbb{P}^{N'}, \quad [z_{N_0}, z_{N_1}, \dots, z_{N_r}] \mapsto [z_{N_1}, \dots, z_{N_r}],$$

which can be resolved by a simple blow-up. By restricting to X, we get the diagram:

$$\begin{array}{ccccc} \tilde{X} = \mathrm{Bl}_Y(X) & \xrightarrow{\ \tilde{\iota}\ } & \widetilde{\mathbb{P}^N} := \mathrm{Bl}_{\mathbb{P}^N_{\text{source}}}(\mathbb{P}^N) & \xrightarrow{\ b\ } & \mathbb{P}^{N'} \\ \downarrow & & \downarrow & & \\ X & \xrightarrow{\ \iota\ } & \mathbb{P}^N & & \end{array} \tag{4.5}$$

If $E \subset \widetilde{\mathbb{P}^N}$ is the exceptional divisor,

$$\mathcal{O}_{\tilde{X}}(E_Y) = \mathcal{O}_{\widetilde{\mathbb{P}^N}}(E)\big|_{\tilde{X}} = \mathcal{O}_b(1) \otimes b^*\mathcal{O}_{\mathbb{P}^{N'}}(-1)\big|_{\tilde{X}},$$

and $\mathcal{O}_b(1)$ is relatively ample. Thus, the requirements of 1.10 are satisfied; to compute the positivity of $Y \subset X$, one must estimate the dimension of $(b\tilde{\iota})(\tilde{X})$. In general, it is not clear how to do this, but we obtain an appealing result under additional hypotheses.

Proposition 4.5 *Let X be a smooth G_m-variety with source Y, and assume that G_m acts on the normal bundle $\mathcal{N}_{Y/X}$ by scalar multiplication. Then $Y \subset X$ is $p^{>0}$, with*

$$p = \dim X - \dim(X \setminus Y^+) - 1.$$

Proof. The first step is to show that $X \subset \mathbb{P}^N$ is invariant under the G_m-action by *scalar multiplication* on the coordinates $(z_{N_1}, \dots, z_{N_r})$. Indeed, since G_m acts by scalar multiplication on $\mathcal{N}_{Y/X}$, it follows that $Y^+ \to Y$ is a vector bundle (cf. [6, Remark pp. 491]), so Y^+ is the total space of $\mathcal{N}_{Y/X}$; for clarity we denote it by $\underline{\mathsf{N}}_{Y/X} := \mathrm{Spec}\left(\mathrm{Sym}^\bullet \mathcal{N}^\vee_{Y/X}\right)$. Thus $\underline{\mathsf{N}}_{Y/X}$ is a Zariski open subset of X, on which G_m-acts by scalar multiplication. Moreover, the inclusions

$$\underline{\mathsf{N}}_{Y/X} \subset \underline{\mathsf{N}}_{\mathbb{P}^N_{\text{source}}/\mathbb{P}^N} \overset{(4.4)}{=} \{\, [z_{N_0} : z_{N_1} : \dots] \mid z_{N_0} \neq 0 \,\} \subset \mathbb{P}^N$$

are G_m-equivariant. But the diagonal multiplication on $(z_{N_1}, \dots, z_{N_r})$ exists on the whole $\mathbb{P}^N$, so $X \subset \mathbb{P}^n$ is G_m-invariant too.

Now we proceed to estimate the dimension of $(b\tilde{\iota})(\tilde{X})$. The previous step implies that for all $\underline{z} = [\underline{z}_0 : \underline{z}'] \in Y^+ \setminus Y$ holds $[\,0 : \underline{z}'\,] = \lim_{t\to\infty} [\,\underline{z}_0 : t \cdot \underline{z}'\,] \in X \setminus Y^+$. This implies that $b(\tilde{X}) = b(X \setminus Y^+)$, as desired. □

4.3. **The Picard group and the diagram** (YSX).

Proposition 4.6 *Let the situation be as in proposition 4.5, with* $\dim X - \dim(X \setminus Y^+) \geqslant 2$. *Then* $\operatorname{Pic}(X) \to \operatorname{Pic}(Y)$ *is an isomorphism.*

Proof. The inclusion $Y^+ \subset X$ yields the exact sequence

$$0 \to \operatorname{Pic}(X \setminus Y^+) = \operatorname{Pic}\Big(\bigcup_{s \neq \text{ source}} \overline{Y_s^+} \Big) \to \operatorname{Pic}(X) \to \operatorname{Pic}(Y^+) \cong \operatorname{Pic}(Y) \to 0. \tag{4.6}$$

The isomorphism on the right hand side holds because $Y^+ \to Y$ is an affine space fibration.

The left hand side is the free abelian group generated by the divisors $\overline{Y_s^+}$, $s \in S_{\mathrm{BB}} \setminus \{\text{source}\}$. But the hypothesis implies that all the components of $X \setminus Y^+$ have codimension at least two in X, so $X \setminus Y^+$ contains no divisors. □

By using the G-action on X, we can deform Y to $Y_g := gY$, with $g \in G$; the latter is the source of the G_m-action for the 1-PS $\lambda_g := \operatorname{Ad}_g(\lambda)$. If G is sufficiently large (see the remark 4.3), the family $\{Y_g\}_{g \in G}$ sweeps out an open subset of X. In order to apply the theorem 2.8, one still has to control the Picard group of the double intersections $Z_g := Y \cap Y_g$. One way to achieve this is by proving sufficient positivity of Y (cf. subsection 4.2.2) and applying theorem 1.18. However, if this method fails, one has to use additional symmetry.

Corollary 4.7 *Let X be a G-variety and λ a 1-PS of G; consider* $\gamma \in \operatorname{Weyl}(G)/\operatorname{Weyl}(G(\lambda))$. *We assume the following:*

(i) *The sources $Y, Y' = \gamma Y$ of λ, λ' intersect transversally; denote $Z := Y \cap Y'$;*
(ii) $\operatorname{codim}_X(Y) + 2 \leqslant \dim X - \dim(X \setminus Y^+)$.

Then, for generic $g \in G$, holds:

- *The intersection $Z_g := Y \cap Y_g$ is transverse.*
- *The restrictions* $\operatorname{Pic}(X) \to \operatorname{Pic}(Y) \to \operatorname{Pic}(Z_g)$ *are isomorphisms.*

Proof. It is enough to prove the claims for $g = \gamma$, because both are unchanged under small perturbations. The isomorphism $\operatorname{Pic}(X) \to \operatorname{Pic}(Y)$ has been proved before. For the isomorphism $\operatorname{Pic}(Y) \to \operatorname{Pic}(Z)$, we observe that λ' leaves Y invariant, so Y admits itself a BB-decomposition. Since $Y \setminus Z = Y \cap (X \setminus Z)$, we deduce that

$$\dim(Y \setminus Z^+) = \operatorname{cd}(Y \setminus Z) \leqslant \operatorname{cd}(X \setminus Z) = \operatorname{cd}(X \setminus Y) = \dim(X \setminus Y^+) \overset{\text{(ii)}}{\leqslant} \dim(Y) - 2.$$

Hence $Y \setminus Z^+$ contains no divisors and the exact sequence (4.6) yields the conclusion. □

Finally, our discussion in this section yields the following splitting criterion.

Theorem 4.8 (i) *Consider the G-subvariety $\mathcal{Y} := \mu(G \times Y) \subset G \times X$. Assume that:*

$$\mathcal{Y} \overset{\rho}{\to} X \text{ is smooth}, \quad \dim X - \dim(X \setminus Y^+) \geqslant 2\operatorname{codim}_X(Y) + 2 \geqslant 6.$$

Then the conditions (YSX) *are satisfied.*
(ii) *Assume furthermore that $Z_g := Y \cap Y_g$ has the property that* $\operatorname{Pic}(Y) \to \operatorname{Pic}(Z_g)$ *is an isomorphism, for $g \in G$ generic. Then the splitting criterion 2.8 applies.*

Note that the condition (ii) can be settled in the situations described in 4.5 and 4.7.

Proof. (i) Since ρ is smooth, so are the generic self intersections of $\mathcal{Y}$. These double and triple self-intersections are also non-empty and connected (cf. 1.15); thus the (1-arm) and (no-$\triangle$) conditions in (YSX) are satisfied.

(ii) By proposition 4.4, we have $\mathrm{cd}(X\setminus Y)=\dim(X\setminus Y^{+})<\dim X-3$. □

Part III: applications

5. Splitting criteria for vector bundles on homogeneous varieties

In this section we specialize the previous discussion to homogeneous varieties. Assume $X=G/P$, where G is connected, reductive, and P is a parabolic subgroup, and consider a 1-PS λ of G. For any parabolic subgroup Q of G, denote $\mathrm{Weyl}(Q):=\mathrm{Weyl}\big(\mathrm{Levi}(Q)\big)$.

The adjoint action of λ on $\mathrm{Lie}(G)$ decomposes it into the direct sum of its weight spaces; we group them into the zero, strictly positive and negative weight spaces:

$$\mathrm{Lie}(G)=\mathrm{Lie}(G)^{0}_{\lambda}\oplus\mathrm{Lie}(G)^{+}_{\lambda}\oplus\mathrm{Lie}(G)^{-}_{\lambda},\ \text{and}\ \mathrm{Lie}(P(\pm\lambda))=\mathrm{Lie}(G)^{0}_{\lambda}\oplus\mathrm{Lie}(G)^{\pm}_{\lambda}. \qquad (5.1)$$

Lemma 5.1 *The following statements hold:*
(i) *The components of X^{λ} are homogeneous for the action of $G(\lambda)$.*
(ii) *The source Y contains $\hat{e}\in G/P$ if and only if $\lambda\subset P$ and $\mathrm{Lie}(G)^{-}_{\lambda}\subset\mathrm{Lie}(P)$.*

Proof. See [13]. □

The maximal torus T acts with isolated fixed points on G/P; they are precisely wP, with $w\in\mathrm{Weyl}(G)/\mathrm{Weyl}(P)$. For any $s\in S_{\mathrm{BB}}$, the component $Y_s\subset X^{\lambda}$ is $G(\lambda)$-invariant and $T\subset G(\lambda)$, so $Y_s^{T}\neq\emptyset$; conversely, λ fixes $(G/P)^{T}$, so $(G/P)^{T}=\bigcup_{s\in S_{\mathrm{BB}}}Y_s^{T}$.

Now we recall some classical facts about Bruhat decompositions (cf. [21, §8], [25, §8]). If $Q,P\subset G$ are two parabolic subgroups, G/P decomposes into locally closed Q-orbits:

$$G/P=\coprod_{w\in S_{\mathrm{Bruhat}}}QwP,\ \text{with}\ S_{\mathrm{Bruhat}}=\mathrm{Weyl}(Q)\big\backslash\mathrm{Weyl}(G)\big/\mathrm{Weyl}(P).$$

Actually, S_{Bruhat} parameterizes the $\mathrm{Weyl}(Q)$-orbits in $(G/P)^{T}$. Each double coset in S_{Bruhat} contains a unique representative of minimal length; for each $w\in S_{\mathrm{Bruhat}}$ of minimal length,

$$\dim(QwP)=\mathrm{length}(w)+\dim\big(\mathrm{Levi}(Q)/\mathrm{Levi}(Q)\cap wPw^{-1}\big).$$

Proposition 5.2 *The Bialynicki-Birula decomposition of G/P for the action of λ coincides with the Bruhat decomposition for the action of $P(\lambda)$.*

Proof. See [13]. □

For homogeneous varieties, the criterion 2.8 yields the following.

Theorem 5.3 *Let $X=G/P$ and λ be a 1-PS of $T\subset G$; we denote by Y its source. Consider also $\gamma\in\mathrm{Weyl}(G)/\mathrm{Weyl}(G(\lambda))$. We assume the following:*
(i) *X is 1-splitting;*
(ii) *$Y, Y'=\gamma Y$ intersect transversally;*
(iii) *$\dim X-\dim(X\setminus Y^{+})\geqslant 2\mathrm{codim}_X(Y)+2\geqslant 6$.*

Then an arbitrary vector bundle $\mathscr{V}$ on G/P splits if and only if $g^{*}\mathscr{V}$ splits on Y, for a very general $g\in G$.

A pleasant feature is that the splitting of $\mathscr{V}$ is reduced to the splitting along a homogeneous subvariety of G/P, so the procedure can be iterated (cf. 5.6 below). Explicit calculations are performed in section 6.

Remark 5.4 The element γ above is typically a simple reflection in $\mathrm{Weyl}(G)\setminus\mathrm{Weyl}(G(\lambda))$. Consider, for instance, $X=\mathrm{Gr}(u;w)$, $G=\mathrm{GL}(w)$, and $\lambda(t)=\mathrm{diag}[t^{-1},1,\dots,1]$. The source of the action is $Y=\{U\subset\mathbb{C}^w\mid e_1=\langle 1,0,\dots,0\rangle\in U\}$. The appropriate γ equals $\mathrm{diag}\left[\begin{bmatrix}0,1\\1,0\end{bmatrix},1,\dots,1\right]$; it is the reflection corresponding to the simple root $\varepsilon_1-\varepsilon_2$.

5.1. **When is G/P a 1-splitting variety?** To settle this question, we need some notations.

$$
\begin{aligned}
\mathcal{X}^*(T) :=&\ \text{the group of characters of } T\text{; similarly for } B,P,G,\text{ etc.;}\\
(\,\cdot\,,\cdot\,)\ &\ \text{the Weyl}(G)\text{-invariant scalar product on } \mathcal{X}^*(T)_{\mathbb{Q}};\\
\Psi :=&\ \text{the roots of } G,\qquad \Delta\subset\Psi \text{ the simple roots;}\\
\Lambda :=&\ \text{the weights } \{\omega\in\mathcal{X}^*(T)_{\mathbb{Q}}\mid (\omega,\beta^\vee)\in\mathbb{Z},\ \forall\,\beta\in\Delta\};\ \beta^\vee:=\tfrac{2\beta}{(\beta,\beta)};\\
&\ \{\omega_\alpha\}_{\alpha\in\Delta} \text{ the fundamental weights, that is } (\omega_\alpha,\beta^\vee)=\mathrm{Kronecker}_{\alpha\beta};\\
\Lambda_+ :=&\ \text{the dominant weights } \{\omega\in\Lambda\mid(\omega,\beta)\geqslant 0,\ \forall\,\beta\in\Delta\};\\
\langle I\rangle :=&\ \text{the vector space generated by } I\subset\Delta;\\
\Lambda(I) :=&\ \Lambda\cap\bigcap_{\alpha\in I}\alpha^\perp=\Lambda\cap\langle I\rangle^\perp,\ \text{the } I\text{-face of } \Lambda.
\end{aligned}
$$

The parabolic subgroup P corresponds to a subset $I\subset\Delta$ (cf. [25, Section 8.4]); we denote it by P_I. Its Weyl group W_I is generated by the reflections τ_α, $\alpha\in I$.

Proposition 5.5 *The homogeneous space $X=G/P_I$ is 1-splitting (cf. 2.9) if and only if there is no simple root perpendicular to $\langle\alpha\mid\alpha\in I\rangle$. Equivalently, define*

$$\tilde I := I\cup\{\beta\in\Delta\setminus I\mid \beta \textit{ is adjacent to some } \alpha\in I \textit{ in the Dynkin diagram of } G\}.$$

Then G/P_I is 1-splitting if and only if $\tilde I=\Delta$. (For Dynkin diagrams, see [25, §9.5]*.)*

Proof. We may assume that G is simply connected, so line bundles on X correspond to characters of P_I. For $\chi\in\mathcal{X}^*(P_I)$, define $\mathcal{L}_\chi:=(G\times\mathbb{C})/P_I$, where $(g,z)\sim(gp^{-1},\chi(p)z)$. By the Borel-Weil-Bott theorem [8, 10], $H^1(G/P_I,\mathcal{L}_\chi)=H^1(G/B,\mathcal{L}_\chi)\neq 0$ if and only if there is $\beta\in\Delta$ and $\chi_+\in\Lambda_+$ such that

$$\chi=\tau_\beta(\chi_++\rho)-\rho=\tau_\beta(\chi_++\beta)\ \Leftrightarrow\ \chi_+=\tau_\beta(\chi+\beta).$$

Let T_β be the transformation $\chi\mapsto\tau_\beta(\chi+\beta)$; it is the reflection in the plane orthogonal to β, passing through $-\frac{\beta}{2}$. The pull-back of $\mathcal{L}_\chi$ to G/B corresponds to the image of χ by $\varpi_I:\mathcal{X}^*(P_I)\to\mathcal{X}^*(B)=\mathcal{X}^*(T)$, so G/P_I is 1-splitting if and only if

$$\Lambda_+\cap T_\beta(\mathrm{Im}(\varpi_I))=\emptyset,\ \forall\,\beta\in\Delta.$$

But $\mathcal{X}^*(P_I)=\mathcal{X}^*(\mathrm{Levi}(P_I))$, so $\mathrm{Im}(\varpi_I)\subset\mathcal{X}^*(T)$ consists of the W_I-invariant elements. Since W_I is generated by the reflections τ_α in the hyperplanes $\alpha^\perp$, $\alpha\in I$, we deduce that $\mathrm{Im}(\varpi_I)=\Lambda(I)$, so we must have

$$\Lambda_+\cap T_\beta\big(\Lambda(I)\big)=\emptyset,\ \forall\,\beta\in\Delta.$$

For $\beta\in I$, this condition is automatically satisfied: $\Lambda(I)\subset\beta^\perp$, so

$$T_\beta\big(\Lambda(I)\big)\subset\{(\beta,\cdot)<0\}\text{ and }\Lambda_+\subset\{(\beta,\cdot)\geqslant 0\}.$$

For $\beta\in\Delta\setminus I$, let $\beta_\perp$ be the component of β on $\langle I\rangle$ with respect to the orthogonal decomposition $\langle\mathcal{X}^*(T)\rangle=\langle I\rangle\oplus\langle I\rangle^\perp$. Then $\beta_\perp$ is also the orthogonal projection of 0 to the affine space $\beta + \langle I\rangle^{\perp}$; hence Λ_+ and $T_\beta\big(\Lambda(I)\big) = \tau_\beta\big(\beta + \Lambda(I)\big)$ are disjoint if and only if they are on different sides of the hyperplane $\langle \tau_\beta \beta_\perp\rangle^{\perp}$:

$$\text{(i) } (\beta_\perp, \beta) > 0, \quad \text{(ii) } (\tau_\beta\beta_\perp, \omega_\alpha) = (\beta_\perp, \tau_\beta\omega_\alpha) \leqslant 0,\ \forall\, \beta \in \Delta \setminus I,\ \forall\, \alpha \in \Delta. \tag{5.2}$$

Claim The inequality (ii) is automatically satisfied.

Case $\alpha \neq \beta$: $\tau_\beta\omega_\alpha = \omega_\alpha$.
– Assume $\alpha \notin I$. Then holds $\omega_\alpha \in \Lambda_+(I) \Rightarrow (\beta_\perp, \omega_\alpha) = 0$.
– Assume $\alpha \in I$. Since any two vectors in I make an angle of at least $90°$ (they are simple roots), and $(-\beta_\perp, c) = -(\beta, c) \geqslant 0$, for all $c \in I$, we deduce that $-\beta_\perp$ is in the cone $\sum_{c\in I} \mathbb{R}_{\geqslant 0} c$. Thus holds:

$$-\beta_\perp = k_\alpha\alpha + \sum_{c\in I\setminus\{\alpha\}} k_c c,\ k_c \geqslant 0 \quad \Rightarrow \quad (\beta_\perp, \omega_\alpha) = -k_\alpha(\alpha, \omega_\alpha) \leqslant 0.$$

Case $\alpha = \beta$: $\omega_\beta \in \Lambda_+(I) \Rightarrow \beta_\perp \perp \omega_\beta \Rightarrow (\beta_\perp, \tau_\beta\omega_\beta) = (\beta_\perp, \omega_\beta - \beta) = -(\beta_\perp, \beta) \leqslant 0$. For the last step: the angle between a vector and its projection to any plane is at most $90°$.

Hence the only relevant condition in (5.2) is the first one. However, $(\beta_\perp, \beta) \geqslant 0$ from the very construction, so we must eliminate the case $(\beta_\perp, \beta) = 0$. This happens precisely when $0 \in \beta + \langle \Lambda(I)\rangle \Leftrightarrow \beta \in \Lambda(I) \Leftrightarrow \beta \perp \alpha,\ \forall\, \alpha \in I$. □

Corollary 5.6 *Assume $T \subset B^- \subset P_I \subset G$, the variety $X = G/P_I$ is 1-splitting, and*

$$2\cdot\#I \geqslant 1 + \#\Delta \quad \textit{(here \# stands for the cardinality).}$$

Then there is $\lambda : G_m \to T$ such that the source Y of the action has the properties:

$$\text{(i) } Y = G(\lambda)/G(\lambda)\cap P_I; \quad \text{(ii) } Y \textit{ is 1-splitting};$$
$$\text{(iii) } G(\lambda)\cap P_I \textit{ corresponds to the simple roots } I\setminus\{\alpha_0\},\ \alpha_0 \in I.$$

Proof. Denote by Ψ_I the roots generated by I; the roots of the unipotent radical $R_u(P_I) \subset P_I$ are $\Psi^- \setminus \Psi_I^-$ (cf. [25, Theorem 8.4.3]). For an arbitrary $\alpha_0 \in I$, we consider a 1-PS λ such that $\langle \alpha_0, \lambda\rangle > 0$, $\langle c, \lambda\rangle = 0$, $\forall\, c \in \Delta\setminus\{\alpha_0\}$. One can easily check the following:

$$\begin{aligned} \text{negative roots of } G(\lambda) &= \Big\{ \sum_{c\in\Delta\setminus\{\alpha_0\}} k_c c \in \Psi \mid k_c \leqslant 0\Big\} \supset \Psi_I^-, \\ \mathrm{Lie}(G)_\lambda^- &= \Big\{ k_0\alpha_0 + \sum_{c\in\Delta\setminus\{\alpha_0\}} k_c c \mid k_0 < 0,\ k_c \leqslant 0 \text{ for } c \neq \alpha_0\Big\} \subset \Psi^-. \end{aligned}$$

It follows that $\mathrm{Lie}(G)_\lambda^- \subset \Psi^- \setminus \Psi_I^-$, which are the roots of $R_u(P_I)$, which implies that $\mathrm{Lie}(G)_\lambda^- \subset \mathrm{Lie}(P_I)$, as desired. It remains to prove that α_0 can be chosen in such a way that Y is 1-splitting, that is:

$$\nexists\, \beta \in \Delta\setminus I \text{ such that } \beta \text{ is adjacent only to } \alpha_0.$$

If such an $\alpha_0 \in I$ does not exist, then for all $\alpha \in I$ there is $\beta \in \Delta\setminus I$ adjacent only to α; this yields an injective function $I \to \Delta\setminus I$, so $\#I \leqslant \#(\Delta\setminus I)$, which contradicts the hypothesis. □

6. Splitting criteria for vector bundles on Grassmannians

The Grassmannian plays a central role because it is a homogeneous variety, and is the 'universal target' for pairs $(X, \mathcal{N})$ consisting of a variety and a globally generated vector bundle on it. Hence the results of both the sections 3, 5 apply. In this section we obtain splitting criteria for vector bundles on the isotropic (symplectic and orthogonal) Grassmannians. The degenerate case, when the bilinear form has kernel, is also included, to demonstrate that theorem 2.8 is not restricted only to the situations discussed in sections 3, 4. All the computations involve two stages: first we compute the positivity of various subvarieties; second, we deduce splitting criteria by restricting vector bundles on the ambient space to them.

Cohomological splitting criteria have been obtained in [22, 19, 3, 18]; however, they involve a large number of conditions. The results below are interesting for their simplicity: indeed, the problem of deciding the splitting of a vector bundle on a Grassmann variety, which is a high dimensional object, is reduced to the splitting along a (very) low dimensional subvariety. Throughout this section, W stands for a $w+1 = \nu+u+1$-dimensional vector space.

6.1. The Grassmannian of linear subspaces.

This case is discussed in [12], where is proved that things are as good as possible, without any genericity assumptions.

Theorem 6.1 *The vector bundle $\mathscr{V}$ on $\mathrm{Gr}(u;\mathbb{C}^w)$, $u \geqslant 2$, $w \geqslant u+2$, splits if and only if its restriction to* an arbitrary $\mathrm{Gr}(2;\mathbb{C}^4) \subset \mathrm{Gr}(u;\mathbb{C}^w)$ *does so.*

This is in perfect analogy with Horrocks' criterion. However, the proof uses a (fortunate) cohomology vanishing, and can not be extended directly. As a warm-up, let us see compute the positivity of a 'smaller' Grassmannian in 'larger' one.

Example 6.2 Let $X = \mathrm{Gr}(u{+}1;\mathbb{C}^{w+1})$ be the Grassmannian of $(u{+}1)$-dimensional subspaces of $\mathbb{C}^{w+1}$. Let $e_0, e_1, \ldots$ be the standard basis of $\mathbb{C}^{w+1}$. The 1-PS

$$\lambda : G_m \to \mathrm{GL}(w+1), \quad \lambda(t) := \mathrm{diag}(t^{-1}, 1, \ldots, 1)$$

acts on X with source $Y = \{U \in X \mid e_0 \in U\} = \mathrm{Gr}(u;\mathbb{C}^{w+1}/\langle e_0\rangle)$. For $U \in Y$, the normal bundle is $\mathcal{N}_{Y/X,U} = \mathrm{Hom}(\langle s\rangle, \mathbb{C}^{w+1}/U)$; since $e_0 \in U$, λ acts trivially on $\mathbb{C}^{w+1}/U$, so it acts by scalar multiplication on the normal bundle. One can easily see that

$$X \setminus Y^+ = \{U \in X \mid U \subset \langle e_1, \ldots, e_w\rangle\} \cong \mathrm{Gr}(u+1; w),$$

so $\mathrm{Gr}(u;\mathbb{C}^w) \subset \mathrm{Gr}(u+1;\mathbb{C}^{w+1})$ is $u^{>0}$ (cf. proposition 4.5).

6.2. The symplectic-isotropic Grassmannian.

Let ω be a skew-symmetric bilinear form on W, such that:

– if $\dim W$ is even, ω is non-degenerate (so ω is a usual symplectic form);

– if $\dim W$ is odd, $\dim \mathrm{Ker}(\omega) = 1$ (ω is symplectic on $W/\mathrm{Ker}(\omega)$ of dimension w).

Let $X := \text{sp-Gr}(u+1;W)$ be the variety of ω-isotropic, $(u+1)$-dimensional subspaces of W. It is a Fano variety with

$$\dim(X) = \frac{(u+1)(2w-3u)}{2}.$$

If $\varphi : \text{sp-Gr}(u+1;W) \to \mathrm{Gr}(u+1;W)$ stands for the natural embedding, then $\mathcal{O}_X(1)$ is the pull-back of the corresponding line bundle on the Grassmannian.

Denote by $G := \mathrm{Sp}_{[(w+1)/2]}$ the symplectic group ($[\,\cdot\,]$ stands for the integral part). If $\dim W$ is even, X is homogeneous for the G-action: $X = G/P$, where P is the stabilizer of the flag

$$\big\{(1,\dots,\underset{u+1}{0},\dots,0 \mid 0,\dots,0),\dots,(0,\dots,\underset{u+1}{1},0,\dots,0 \mid 0,\dots,0)\big\} \subset \mathbb{C}^{\frac{w+1}{2}} \oplus \mathbb{C}^{\frac{w+1}{2}}.$$

If $\dim W$ is odd, X has two G-orbits: the open orbit of subspaces which intersect $\mathrm{Ker}(\omega)$ trivially, and the closed orbit of subspaces containing $\mathrm{Ker}(\omega)$.

Lemma 6.3 *If* $w \geqslant 2u + 1 + \dim \mathrm{Ker}(\omega)$, *then* $\mathrm{Pic}(X) = \mathbb{Z} \cdot \mathcal{O}_X(1)$.

Proof. For ω non-degenerate, this is clear. For $\mathrm{Ker}(\omega) = \langle s_0\rangle$,

$$\text{sp-Gr}(u+1; W) = \{U \mid s_0 \in W\} \cup \{U \mid s_0 \notin U\}.$$

The first term is a subvariety, isomorphic to $\text{sp-Gr}(u; W/\langle s_0\rangle)$, of codimension $w - 2u \geqslant 2$; thus $\mathrm{Pic}(X)$ is isomorphic to the Picard group of the open stratum. The morphism

$$\{U \mid s_0 \notin U\} \to \text{sp-Gr}(u+1, W/\langle s_0\rangle), \quad [U \subset W] \mapsto \big[\langle s_0\rangle + U/\langle s_0\rangle \subset W/\langle s_0\rangle\big]$$

is an affine space fibration, and the base has cyclic Picard group. $\square$

The quotient bundle $\mathcal{N}$ and the tautological bundle $\mathcal{U} := \mathrm{Ker}(W \otimes \mathcal{O}_X \to \mathcal{N})$ on X are the pull-back by φ of their counterparts on the Grassmannian. An element $s \in W \setminus \mathrm{Ker}(\omega)$ defines a section in $\mathcal{N}$, whose zero set is the 'smaller' isotropic Grassmannian:

$$\begin{aligned}\{U \in \text{sp-Gr}(u; W) \mid s \in U\} &= \{U \in \text{sp-Gr}(u; W) \mid s \in U \subset U^\perp \subset \langle s\rangle^\perp\}\\ &= \text{sp-Gr}(u; \langle s\rangle^\perp/\langle s\rangle) = \text{sp-Gr}(u+1; W) \cap \mathrm{Gr}(u; W/\langle s\rangle), \end{aligned} \tag{6.1}$$

where $\langle s\rangle^\perp := \{t \in W \mid \omega(s,t) = 0\}$.

An element $\sigma \in W^\vee \setminus \{0\}$ determines a section in $\mathcal{U}^\vee$, with zero locus

$$\text{sp-Gr}(u+1, \sigma^\perp) = \{U \in \text{sp-Gr}(u+1; W) \mid U \subset \sigma^\perp\}, \text{ where } \sigma^\perp := \mathrm{Ker}(\sigma). \tag{6.2}$$

In particular, $s \in \big(W/\mathrm{Ker}(\omega)\big) \setminus \{0\}$ determines $\sigma_s(\cdot) := \omega(s,\cdot) \in W^\vee$; in this case, $\sigma_s^\perp = \langle s\rangle^\perp$. For ω non-degenerate, $s \mapsto \sigma_s$ defines an isomorphism $W \to W^\vee$; however, if $\dim \mathrm{Ker}(\omega) = 1$, the image of this map is a hyperplane in $W^\vee$. In the latter case, for generic $\sigma \in W^\vee$, $\omega|_{\sigma^\perp}$ is non-degenerate, so $\sigma^\perp$ is a symplectic subspace of W. In general, it holds

$$\dim \mathrm{Ker}(\omega|_{\sigma^\perp}) = 1 - \dim \mathrm{Ker}(\omega), \text{ for generic } \sigma. \tag{6.3}$$

We start by explicitly computing the positivity of some subvarieties of X. Deliberately, we consider both zero loci of sections and sources of G_m-actions, to illustrate the general theory developed in the previous sections.

Lemma 6.4 (i) *For* $s \in W^\vee \setminus \{0\}$ *and* $u \geqslant 1$,

$$\text{sp-Gr}(u+1; s^\perp) \subset \text{sp-Gr}(u+1; W) \textit{ is } (w - 2u - 1)^{>0}. \quad (\textit{Recall that } s^\perp := \mathrm{Ker}(s).)$$

(ii) *Assume that either* $\mathrm{Ker}(\omega) = \langle s\rangle \subset W$, *or* ω *is non-degenerate and* $s \neq 0$. *Then*

$$\text{sp-Gr}(u; \langle s\rangle^\perp/\langle s\rangle) \subset \text{sp-Gr}(u+1; W) \textit{ is } u^{>0}, \textit{ for } u \geqslant 1.$$

(iii) *Decompose* $W = W' \oplus W''$ *into the sum of Lagrangian subspaces of dimension* $(w+1)/2$. *Then* $\mathrm{Gr}(u+1; W') \subset \text{sp-Gr}(u+1, W)$ *is* $\big(\frac{w-2u-1}{2}\big)^{>0}$.

Proof. (i) In the diagram

$$\begin{array}{ccc} \text{sp-Gr}(u+1;W) & \xrightarrow[U\mapsto U\cap s^\perp]{b} & \text{sp-Gr}(u;s^\perp) \\ \downarrow\varphi & & \downarrow\varphi \\ \text{Gr}(u+1;W) & \xrightarrow[U\mapsto U\cap s^\perp]{g} & \text{Gr}(u;s^\perp), \end{array} \tag{6.4}$$

g is undefined on $\text{Gr}(u+1;s^\perp)$, and b is undefined on $\{U \mid U \subset s^\perp\} = \text{sp-Gr}(u+1;s^\perp)$. The blow-up of $\text{Gr}(u+1;W)$ along $\text{Gr}(u+1;s^\perp)$ resolves g, hence b. It remains to apply proposition 3.3: the fibres of b are at least $\frac{(u+1)(2w-3u)}{2} - \frac{u(2w-3u+1)}{2} = w-2u$ dimensional.
(ii) *Case* $\text{Ker}(\omega) = \langle s\rangle$. In the diagram

$$\begin{array}{ccc} \text{sp-Gr}(u+1;W) & \xrightarrow[U\mapsto(\langle s\rangle+U)/\langle s\rangle]{b} & \text{sp-Gr}(u+1;W/\langle s\rangle) \\ \downarrow\varphi & & \downarrow\varphi \\ \text{Gr}(u+1;W) & \xrightarrow[U\mapsto(\langle s\rangle+U)/\langle s\rangle]{g} & \text{Gr}(u+1;W/\langle s\rangle). \end{array} \quad \Big(\tfrac{\langle s\rangle+U}{\langle s\rangle} \subset \tfrac{W}{\langle s\rangle} \text{ is isotropic.}\Big) \tag{6.5}$$

b is not defined on $\text{Gr}(u;W/\langle s\rangle) \cap \text{sp-Gr}(u+1;W) = \text{sp-Gr}(u;W/\langle s\rangle)$. The blow-up of $\text{Gr}(u;W/\langle s\rangle)$ resolves g, hence b. Now apply 3.3 again: the general fibre of b is $(u+1)$-dimensional.
Case W is symplectic. To settle this, we use the proposition 1.4 in conjunction with 4.4. Decompose W into a direct sum of Lagrangian subspaces $W = \mathbb{C}^{(w+1)/2}_{\text{left}} \oplus \mathbb{C}^{(w+1)/2}_{\text{right}}$, and assume that $s = (1,\dots,0\,|\,0,\dots,0)$. Then $Y := \text{sp-Gr}(u, \langle s\rangle^\perp/\langle s\rangle)$ is the source of the G_m-action, corresponding to the 1-PS:

$$\lambda : G_m \to \text{Sp}_{(w+1)/2}(\mathbb{C}), \quad \lambda(t) = \text{diag}\left[t^{-1}, \mathbb{1}_{(w-1)/2}, t, \mathbb{1}_{(w-1)/2}\right] \tag{6.6}$$

The complement of the open BB-cell is $X \setminus Y^+ = \big\{U \in X \mid s \notin \lim_{t\to 0}\lambda(t)U\big\}$. Consider a basis in U such that the corresponding column matrix is lower triangular; then one sees that $X \setminus Y^+ = \{U \mid U \subset \mathbb{C}^{(w-1)/2}_{\text{left}} \oplus \mathbb{C}^{(w+1)/2}_{\text{right}}\}$, which is $\frac{(u+1)(2(w-1)-3u)}{2}$-dimensional, so

$$\text{cd}(X \setminus Y) = \dim X - (u+1).$$

It remain to compute the amplitude of the normal bundle $\mathcal{N}_{Y/X}$, which is isomorphic to the universal quotient bundle $\mathcal{N}$ on Y. This is globally generated, so we can use the criterion 3.1 to compute its ampleness. The homomorphism $\frac{W}{\langle s\rangle} \otimes \mathcal{O}_Y \to \mathcal{N}$ induces the surjective map $\mathbb{P}(\mathcal{N}^\vee) \to \mathbb{P}^{w-1}$. By homogeneity, its fibres are isomorphic, hence $\mathcal{N}_{Y/X}$ is $(\dim Y - u)$-ample.
(iii) We are going to apply the proposition 4.5. For a Lagrangian direct sum decomposition $W = W' \oplus W'' = \mathbb{C}^{(w+1)/2}_{\text{left}} \oplus \mathbb{C}^{(w+1)/2}_{\text{right}}$, we observe that $Y := \text{Gr}(u+1, \mathbb{C}^{(w+1)/2}_{\text{left}})$ is the source of the G_m-action

$$\lambda : G_m \to \text{Sp}_{(w+1)/2}(\mathbb{C}), \quad \lambda(t) = \text{diag}\left[t^{-1}\mathbb{1}_{(w+1)/2}, t\mathbb{1}_{(w+1)/2}\right]. \tag{6.7}$$

An easy computation yields:

$$\begin{aligned} T_{X,U} &= \{h \in \text{Hom}\big(U, W/U\big) \mid \omega(u', hu'') + \omega(hu', u'') = 0, \forall\, u', u'' \in U\} \\ &\cong \text{Hom}(U, U^\perp/U) \oplus \text{Hom}^{\text{symm}}(U, U^\vee), \\ T_{Y,U} &= \text{Hom}(U, W'/U), \\ \mathcal{N}_{Y/X,U} &= \text{Hom}(U, U^\perp/W') \oplus \text{Hom}^{\text{symm}}(U, U^\vee). \end{aligned}$$

On both summands λ acts diagonally, with weight t^2. Furthermore, one can see that

$$X \setminus Y^+ = \{U \mid \mathrm{Ker}(\mathrm{pr} : U \to \mathbb{C}^{(w+1)/2}_{\mathrm{left}}) \neq 0\}.$$

The minimal degeneration is when Ker(pr) is one dimensional; the corresponding stratum maps onto $\mathbb{P}(\mathbb{C}^{(w+1)/2}_{\mathrm{right}})$, with fibres isomorphic to $\text{sp-Gr}(u; \mathbb{C}^{w-1})$. It follows that $\dim X - \dim(X \setminus Y^+) = \frac{w-2u+1}{2}$. □

Remark 6.5 (i) Sommese's criterion 3.1 implies that the quotient bundle on $\text{sp-Gr}(u+1; W)$ is q-ample, for $q = \dim \text{sp-Gr}(u+1; W) - (u+1)$. Hence $\mathrm{Gr}(u; \langle s\rangle^\perp/\langle s\rangle)$ is $p^{>0}$, with

$$p = (u+1) - \operatorname{codim} \text{sp-Gr}(u; \langle s\rangle^\perp/\langle s\rangle) = 2u - w + 1 < 0.$$

This shows that this test is week compared with proposition 3.3.
(ii) The conclusion of 6.4(i), for ω non-degenerate, can not be obtained by using a 1-PS of G, because $\text{sp-Gr}(u+1; s^\perp)$ is not homogeneous. In contrast, it is not clear how to prove 6.4(iii) by using the proposition 3.3.
(iii) At 6.4(ii), for ω degenerate, the section s (with $\langle s\rangle = \mathrm{Ker}(\omega)$) is *neither generic nor regular* (transverse to zero); thus we really use the generality in the subsection 3.2.

For ω non-degenerate, the computation 6.4(ii) is one of the most technical ones. None of the other methods (propositions 3.3 and 4.5) yield the $u^{>0}$ positivity. Proposition 1.4 readily implies that $\text{sp-Gr}(u; w-1) \subset \text{sp-Gr}(u+1; w+1)$ is $\min(u, w-2u-1)^{\gtrsim 0}$, but this is too weak to obtain splitting criterion for $\text{sp-Gr}(u; 2u)$.

Proposition 6.6 *Let ω be a skew-symmetric bilinear form on W, $\dim W = w+1$, and $\kappa := \dim \mathrm{Ker}(\omega) \leqslant 1$. Let $X := \text{sp-Gr}(u+1; W)$, and $\mathscr{V}$ be an arbitrary vector bundle on X. In the cases enumerated below, $\mathscr{V}$ splits if and only if $\mathscr{V}_{Y_s}$ splits.*
(i) *$Y_s := \text{sp-Gr}(u+1; s^\perp)$, with $s \in W^\vee$ very general, and*

$$w + 1 > 2(u+1) + \kappa, \text{ so } w \geqslant 2u + 3 + \kappa, \text{ and } u \geqslant 1.$$

(ii) *$Y_s := \text{sp-Gr}(u; \langle s\rangle^\perp/\langle s\rangle)$, with $s \in W \setminus \mathrm{Ker}(\omega)$ very general, and $w \geqslant 2u + 1 + \kappa$, $u \geqslant 2$.*

Proof. In both cases, we verify (YSX) and apply theorem 2.8 directly.
(i) Take $S \subset W^\vee \setminus \{0\}$ open subset such that (6.3) holds, and $\mathcal{Y} \subset S \times X$ be the zero locus of the universal section in $(\mathrm{pr}^{S\times X}_X)^* \mathcal{U}^\vee$. The morphisms $\mathcal{Y} \xrightarrow{\pi} S$ and $\mathcal{Y} \xrightarrow{\rho} X$ are both open, and $\pi^{-1}(s) = Y_s$ is projective, connected; also, $\dim \mathrm{Ker}(\omega|_{s^\perp}) = 1 - \kappa$.
– We claim that, for all $o, s, t \in S$,

$$Y_{os} = \{U \in X \mid U \subset \langle o, s\rangle^\perp\} \cong \text{sp-Gr}(u+1; \langle o, s\rangle^\perp), \text{ and}$$
$$Y_{ost} = \text{sp-Gr}(u+1; \langle o, s, t\rangle^\perp) \cong \text{sp-Gr}(u+1; \langle o, s, t\rangle^\perp)$$

are connected and non-empty. The connectedness is clear, since they are quasi-homogeneous, with finitely many orbits, for actions of appropriate subgroups of $\mathrm{Sp}(W)$. We verify that $Y_{ost} \neq \emptyset$ (thus (1-arm) and (no-△) are satisfied), that is

$$\exists\, U \in X \text{ such that } U \subset \langle o, s, t\rangle^\perp.$$

Let ω_{ost} be the restriction of ω to $\langle o, s, t\rangle^\perp$; then $\dim \mathrm{Ker}(\omega_{ost}) = 1 - \kappa$. We verify:

$$\dim \langle o, s, t\rangle^\perp / \mathrm{Ker}(\omega_{ost}) \geqslant 2\big((u+1) - \dim \mathrm{Ker}(\omega_{ost})\big) \Leftrightarrow w + \dim \mathrm{Ker}(\omega_{ost}) \geqslant 2u + 4.$$

The last inequality is fulfilled, by hypothesis.

– The diagram (YSX)(Pic) consists of isomorphisms (cf. lemma 6.3).
– Finally, $Y_s \subset X$ is $(w-2u-1)^{>0}$ (cf. lemma 6.4(i)), so at least $2^{>0}$.

(ii) Let $S := W \setminus \mathrm{Ker}(\omega)$, and $\mathcal{Y}$ be the zero locus of the universal section in $(\mathrm{pr}_X^{S\times X})^*\mathcal{N}$. The situation is analogous, with a few differences. Indeed, for $o, s \in S$ holds:

$$Y_{os} \neq \emptyset \ \Leftrightarrow\ o \perp s, \text{ so generically } Y_{os} = \emptyset.$$

For $o \in S$, we check the properties (YSX) for

$$S(o) = \{s \in S \mid Y_{os} \neq \emptyset\} = S \cap \langle o\rangle^{\perp}.$$

– The diagram (Pic) consists of isomorphisms (cf. lemma 6.3).
– The condition (1-arm), that is $\rho(\mathcal{Y}_{S(o)}) = X$, is the following:

$$\forall U \in X\ \exists s \in S \cap \langle o\rangle^{\perp}\ \exists V \in X \text{ such that } s \in U,\ \langle o, s\rangle \subset V.$$

Indeed, $\dim(U \cap \langle o\rangle^{\perp}) \geqslant u$, so $U \cap \langle o\rangle^{\perp} \neq 0$. Take $s \in (U \cap \langle o\rangle^{\perp}) \setminus \mathrm{Ker}(\omega)$, non-zero; then $\langle o, s\rangle \subset W$ is an isotropic subspace. There exists $V \in X$ containing it, as $u \geqslant 2$.
– The condition (no-$\triangle$) reads:

$$[s \perp o,\ t \perp o,\ s \perp t] \ \Rightarrow\ \exists V \in X,\ \langle o, s, t\rangle \subset V.$$

The left hand side implies that $\langle o, s, t\rangle \subset W$ is isotropic subspace; then V exists, since $u \geqslant 2$.
– Finally, by lemma 6.4(ii), $Y_s \subset X$ is $u^{>0}$, so at least $2^{\gtrsim 0}$. □

Theorem 6.7 *Let ω be a skew-symmetric bilinear form on $\mathbb{C}^w$, with $\kappa = \dim \mathrm{Ker}(\omega) \leqslant 1$. We consider the isotropic Grassmannian $X = \text{sp-Gr}(u; \mathbb{C}^w)$, with $u \geqslant 2$, and an arbitrary vector bundle $\mathscr{V}$ on it. Then $\mathscr{V}$ splits if and only if it does so along a very general subvariety $Y \cong \text{sp-Gr}(2; 4+\kappa)$ of X.*

Note that sp-Gr(2; 4), the Lagrangian 2-planes in $\mathbb{C}^4$, is isomorphic through the Plücker embedding to the 3-dimensional quadric.

Proof. By applying repeatedly the first part of previous proposition (after replacing $w{+}1 \rightsquigarrow w$ and $u+1 \rightsquigarrow u$), we deduce that $\mathscr{V}$ splits if and only if $\mathscr{V}_Z$ splits, for some very general subvariety $Z \cong \text{sp-Gr}(u, 2u+\kappa)$ of X. Now apply the second part to deduce reduce the splitting problem from Z to $Y \cong \text{sp-Gr}(2, 4+\kappa)$. (For this latter, the process can not be iterated anymore.) □

6.3. **The orthogonal-isotropic Grassmannian.** The situation is similar to the previous case: let β be a symmetric, non-degenerate, bilinear symmetric form on W, and consider $X := \text{o-Gr}(u+1; W)$ be the variety of isotropic $(u+1)$-dimensional subspaces of W; assume $u \geqslant 1$, $\dim W \geqslant 5$. (If $w+1 = 2(u+1)$, the full space of Lagrangian planes in $\mathbb{C}^{2(u+1)}$ has two connected components, and we consider only one of them.) It is a homogeneous variety for $G = \mathrm{SO}(\beta)$, with

$$\dim X = \frac{(u+1)(2w-3u-2)}{2}, \text{ and } \mathrm{Pic}(X) = \mathbb{Z}\cdot \mathcal{O}_X(1).$$

Similar arguments as before yield the following.

Lemma 6.8 (i) *If $u \geqslant 1$, $w \geqslant 2u+2$, then*

$$\text{o-Gr}(u+1; \langle s\rangle^{\perp}) \subset \text{o-Gr}(u+1; W) \text{ is } (w-2u-2)^{>0}.$$

The form $\beta|_{\langle s\rangle^{\perp}}$ is non-degenerate if and only if s is a non-isotropic vector.

(ii) *Let* $s \in W$ *be isotropic,* $u \geqslant 2$. *Then* $\text{o-Gr}(u; \langle s\rangle^{\perp}/\langle s\rangle) \subset \text{o-Gr}(u+1; W)$ *is*

$$\begin{cases} (u-1)^{>0} & \textit{for } w = 2u+1, \\ u^{>0} & \textit{for } w \geqslant 2u+2. \end{cases}$$

Proof. (i) We repeat the argument in the lemma 6.4(i), and use that $\dim \text{o-Gr}(u+1; W) - \dim \text{o-Gr}(u; \langle s\rangle^{\perp}) = w - 2u - 1$.
(ii) We are going to apply the proposition 4.5. Decompose $W = \mathbb{C}^{(w+1)/2} \oplus \mathbb{C}^{(w+1)/2}$ into the sum of two Lagrangian subspaces and consider the 1-PS:

$$\lambda : G_m \to \mathrm{SO}_{(w+1)/2}, \quad \lambda(t) = \big[t^{-1}, \mathbb{1}_{(w-1)/2}, t, \mathbb{1}_{(w+1)/2}\big].$$

The source of λ is $Y = \{U \mid s := (1, 0, \dots, 0) \in U\}$, and a simple computation shows that, for all $U \in Y$, holds:

$$\begin{aligned} T_{X,U} &= \{h \in \mathrm{Hom}\big(U, W/U\big) \mid \beta(u', hu'') + \beta(hu', u'') = 0, \forall\, u', u'' \in U\} \\ &\cong \mathrm{Hom}(U, U^{\perp}/U) \oplus \mathrm{Hom}^{\text{anti-symm}}(U, U^{\vee}), \qquad (\text{Note that } h(s) \in \langle s\rangle^{\perp}.) \\ T_{Y,U} &\cong \mathrm{Hom}(U/\langle s\rangle, U^{\perp}/U) \oplus \mathrm{Hom}^{\text{anti-symm}}\big(U/\langle s\rangle, (U/\langle s\rangle)^{\vee}\big), \\ \mathcal{N}_{Y/X,U} &= \mathrm{Hom}(\langle s\rangle, \langle s\rangle^{\perp}/U). \end{aligned}$$

It follows that λ acts by scalar multiplication on $\mathcal{N}_{Y/X}$, with weight t. Furthermore, the complement of the open BB-cell is $X \setminus Y^{+} = \big\{U \in X \mid s \notin \lim\limits_{t\to 0} \lambda(t)U\big\}$. By taking a basis in U such that the corresponding column matrix is lower triangular, one can see that

$$X \setminus Y^{+} = \{U \mid U \subset W' := \mathbb{C}^{(w-1)/2} \oplus \mathbb{C}^{(w+1)/2}\}.$$

Note that $\beta|_{W'}$ has a 1-dimensional kernel; denote it by $\langle s'\rangle$. If $w \in \{2u+1, 2u+2\}$, one can see that $s' \in U$ for all $U \in X \setminus Y^{+}$. By using this remark one finds that $\dim X - \dim(X \setminus Y^{+})$ equals: u for $w = 2u+1$, and $u+1$ for $w \geqslant 2u+2$. □

Proposition 6.9 *Let* β *be a non-degenerate symmetric bilinear form on* W*, with* $\dim W = w+1$. *Let* $X := \text{o-Gr}(u+1; W)$*, and* $\mathscr{V}$ *be an arbitrary vector bundle on* X. *In the cases enumerated below,* $\mathscr{V}$ *splits if and only if* $\mathscr{V}_{Y_s}$ *splits.*
(i) $Y_s := \text{o-Gr}(u+1; \langle s\rangle^{\perp})$*, with* $s \in W$ *very general, non-isotropic, and* $w \geqslant 2u+4$.
(ii) $Y_s := \text{o-Gr}(u; \langle s\rangle^{\perp}/\langle s\rangle)$*, with* s *very general isotropic, and* $w \geqslant 2u+1$*,* $u \geqslant 3$.

Proof. Again we check (YSX) and apply 2.8 directly. Let $Q_\beta := \{s \in W \mid \beta(s,s) = 0\}$ be the isotropic cone.
(i) Take $S := W \setminus Q_\beta$ and let $\mathcal{Y} \subset S \times X$ be the universal family, with $Y_s = \text{o-Gr}(u+1, \langle s\rangle^{\perp})$.
– For $o, s, t \in S$,

$$Y_{os} = \{U \in X \mid U \subset \langle o, s\rangle^{\perp}\}, \quad Y_{ost} = \{U \in X \mid U \subset \langle o, s, t\rangle^{\perp}\}.$$

They are quasi-homogeneous for the action of appropriate subgroups of $\mathrm{SO}(\beta)$, thus connected. As $w \geqslant 2u+4$, we deduce $\dim \langle o, s, t\rangle^{\perp} \geqslant 2 \dim U$, so Y_{ost} is always non-empty; in particular, (1-arm) and (no-△) are satisfied.
– Finally, the diagram (Pic) consists of isomorphisms.
(ii) Here we choose $S := Q_\beta \setminus \{0\}$; the situation is similar to 6.6(ii). For $o, s \in S$,

$$Y_{os} \neq \emptyset \;\Leftrightarrow\; s \in \langle o\rangle^{\perp} \cap S; \quad \text{let } S(o) := \langle o\rangle^{\perp} \cap S.$$

We check the conditions (YSX) for $\mathcal{Y}_{S(o)}$.

– $W_{os} := \langle o,s\rangle^{\perp}/\langle o,s\rangle$ has an induced non-degenerate orthogonal form, so Y_{os} is connected.
– The diagram (Pic) consists of isomorphisms.
– The condition (1-arm), that is $\rho(\mathcal{Y}_{S(o)}) = X$ is:

$$\forall U \in X\ \exists s \in S(o)\ \exists V \in X \text{ such that } s \in U,\ \langle o,s\rangle \subset V.$$

Indeed, take $s \in U \cap \langle o\rangle^{\perp}$ non-zero, so $\langle o,s\rangle \subset W$ is isotropic; now take any V containing it.
– The condition (no-$\triangle$) reads: $\big[o \perp s,\ o \perp t,\ s \perp t\big] \Rightarrow Y_{ost} \neq \emptyset$.
Indeed, $\langle o,s,t\rangle \subset W$ is isotropic, so there is $U \in X$ containing it because $u+1 \geqslant 3$.
– By lemma 6.8(ii), $Y_s \subset X$ is at least $2^{>0}$. □

Theorem 6.10 *Let ω be a non-degenerate bilinear form on $\mathbb{C}^w$. We consider the isotropic Grassmannian $X = \text{sp-Gr}(u;\mathbb{C}^w)$, with $u \geqslant 3$, and an arbitrary vector bundle $\mathscr{V}$ on it. Then $\mathscr{V}$ splits if and only if it $\mathscr{V}_Y$ splits, with Y very general, where:*

- *$Y \cong \text{o-Gr}(3,\mathbb{C}^6)$, if $w = 2u$;*
- *$Y \cong \text{o-Gr}(3,\mathbb{C}^7)$, if $w = 2u+1$;*
- *$Y \cong \text{o-Gr}(3,\mathbb{C}^8)$, if $w \geqslant 2u+2$.*

Proof. Assume that $w \geqslant 2u+2$. Then the first part of previous proposition implies (after replacing $w+1 \rightsquigarrow w$ and $u+1 \rightsquigarrow u$) that $\mathscr{V}$ splits if and only if $\mathscr{V}_Z$ splits, for some very general subvariety $Z \cong \text{sp-Gr}(u, 2u+2)$ of X. There remain three possibilities: $\text{o-Gr}(u,2u), \text{o-Gr}(u,2u+1), \text{o-Gr}(u,2u+2)$. The theorem follows now from the second part of the proposition. □

The somewhat non-uniform formulation of the theorem, compared to 6.7, is due to the lack of sufficient positivity of $Y = \text{o-Gr}(u;2u+1) \subset \text{o-Gr}(u,2u+2) = X$, which is only $1^{>0}$.

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E-mail address: `mihai.halic@gmail.com`